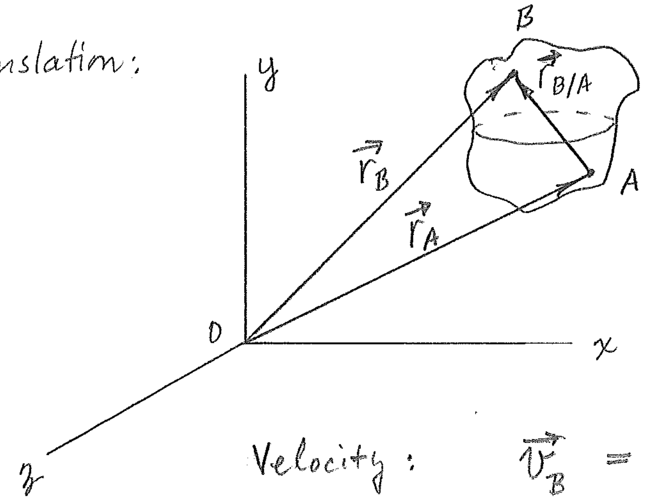
Rigid Body Dynamics

1. Kinematics

General Motion of Rigid Bodies may be Decomposed into Translation + Rotation.

Translation:



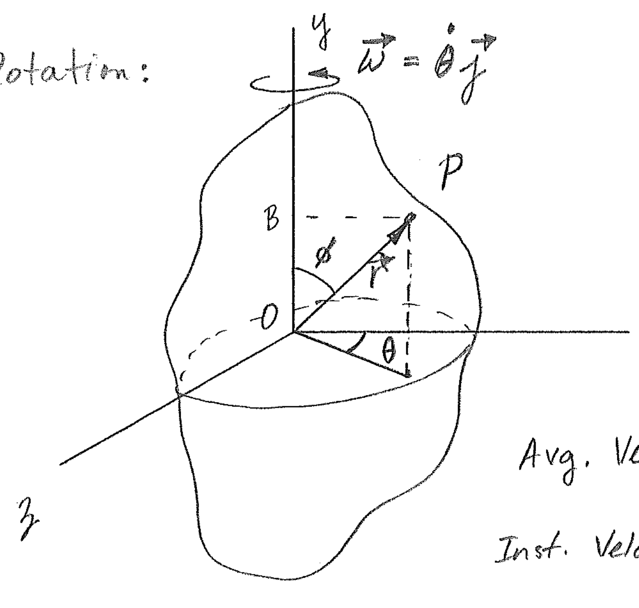
$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

Velocity: $$\vec{v}_B = \frac{d\vec{r}_B}{dt} = \underbrace{\frac{d\vec{r}_A}{dt}}_{=\vec{v}_A} + \underbrace{\frac{d\vec{r}_{B/A}}{dt}}_{=0}$$

$$\therefore \vec{v}_B = \vec{v}_A$$

Similarly, $$\vec{a}_B = \vec{a}_A$$

Rotation:



For Circular Motion

Let Δs = Displacement on Circular Arc.

$$\Delta s = (BP)\,\Delta\theta$$
$$= (r\sin\phi)\,\Delta\theta$$

Avg. Velocity: $$\frac{\Delta s}{\Delta t} = (r\sin\phi)\frac{\Delta\theta}{\Delta t}$$

Inst. Velocity: $$v = \frac{ds}{dt} = \underbrace{r\dot{\theta}\sin\phi}_{\text{cross product}}$$

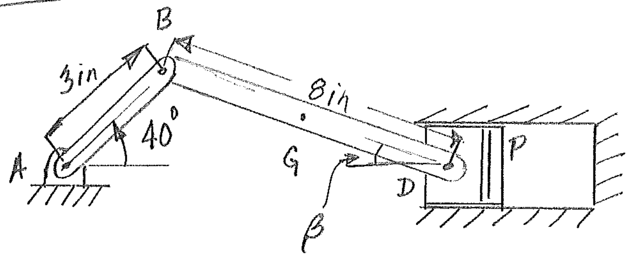
$$\therefore \vec{v} = \vec{\omega}\times\vec{r}$$

Acceleration: $$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega}\times\vec{r})$$

$$= \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt}$$

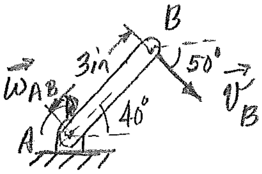
$$\therefore \vec{a} = \vec{\alpha}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

Example :



Crank AB has a constant clockwise angular velocity of 2000 rpm. For the position shown; determine, (a) $\vec{\omega}_{BD}$ (b) $\vec{v}_P$

Consider crank AB first

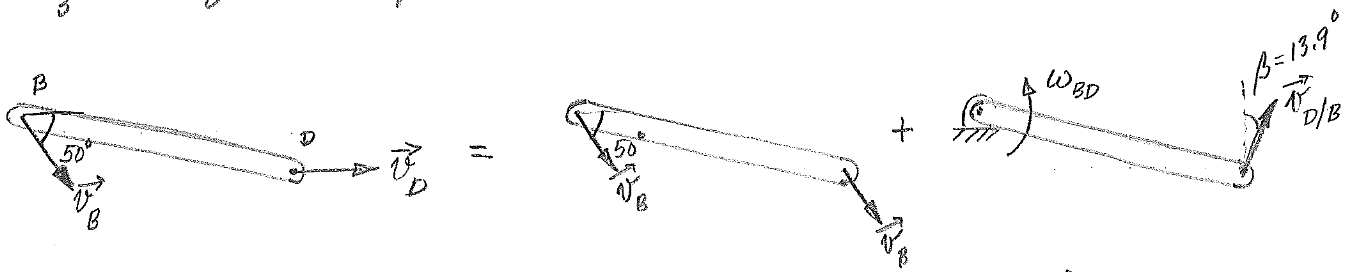


$$\omega_{AB} = \left(2000\ \frac{rev}{min}\right)\left(\frac{1\ min}{60\ sec}\right)\left(2\pi\ \frac{rad}{rev}\right) = 209\ \frac{rad}{sec}$$

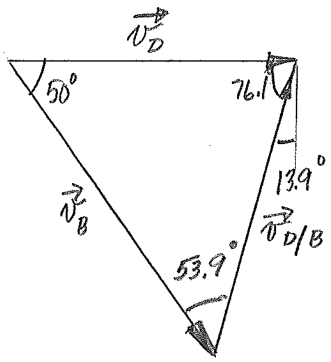
$$v_B = (AB)\,\omega_{AB} = (3in)\left(209\ \frac{rad}{sec}\right) = 627\ \frac{in}{sec} \qquad \vec{v}_B = 627\ \frac{in}{sec}\ \measuredangle 50^\circ$$

Motion of Connecting Rod is General Plane Motion. Determine angle β

$$\frac{\sin\beta}{3} = \frac{\sin 40^\circ}{8} \implies \beta = 13.9^\circ$$



$$\vec{v}_D = \vec{v}_B + \vec{v}_{D/B}$$



$$\frac{v_D}{\sin 53.9^\circ} = \frac{v_B}{\sin 76.1^\circ} = \frac{v_{D/B}}{\sin 50^\circ}$$

$$v_D = \frac{\sin 53.9^\circ}{\sin 76.1^\circ}\left(627\ \frac{in}{sec}\right) \implies \vec{v}_D = 522\ \frac{in}{sec} \longrightarrow$$

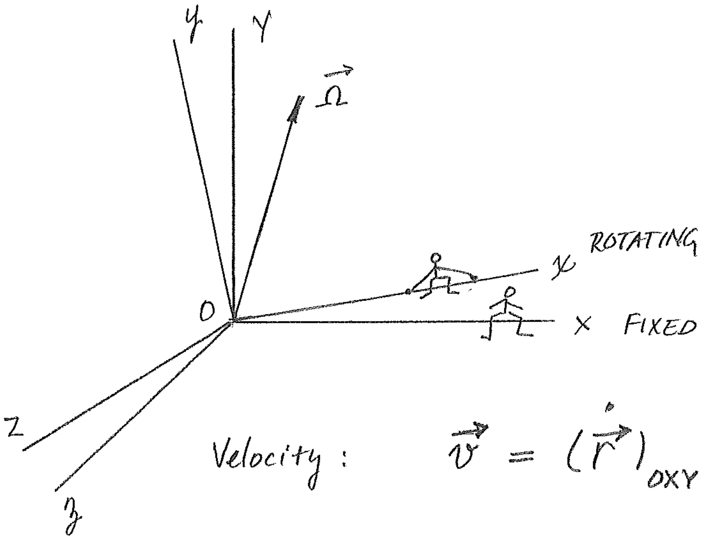
$$v_{D/B} = \frac{\sin 50^\circ}{\sin 76.1}\left(627\ \frac{in}{sec}\right) \implies \vec{v}_{D/B} = 495\ \frac{in}{sec}\ \measuredangle 76.1^\circ$$

Since $v_{D/B}$ is Rotation,

$$v_{D/B} = l \cdot \omega_{BD}$$

$$\omega_{BD} = \frac{495\ \frac{in}{sec}}{8\ in} \implies \vec{\omega}_{BD} = 61.9\ \frac{rad}{sec}$$

Rotating Frame of Reference



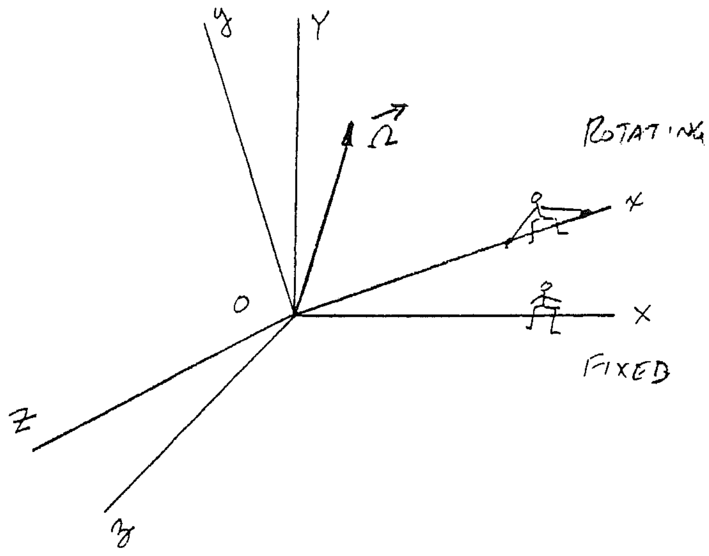
Rotating Frame of Reference Oxyz Turns with Angular Velocity $\vec{\Omega}$.

Velocity:
$$\vec{v} = (\dot{\vec{r}})_{OXYZ} = \underbrace{\vec{\Omega}\times\vec{r}}_{\text{ENTRAINED TERM}} + \underbrace{(\dot{\vec{r}})_{Oxyz}}_{\text{RELATIVE TERM}}$$

The Absolute Velocity is equal to the Velocity with respect to the Rotating Oxyz plus the Entrained Term, which includes the effect of Oxyz's Rotation.

Acceleration:
$$\vec{a} = \dot{\vec{v}} = \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times\dot{\vec{r}} + \frac{d}{dt}(\dot{\vec{r}})_{Oxyz}$$
$$= \underbrace{\vec{\alpha}\times\vec{r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})}_{\text{ENTRAINED TERMS}} + \underbrace{(\ddot{\vec{r}})_{Oxyz}}_{\text{RELATIVE TERM}} + \underbrace{2\vec{\Omega}\times\dot{\vec{r}}_{Oxyz}}_{\text{CORIOLIS TERM}}$$

The Absolute Acceleration is equal to the Relative Acceleration within the Roting Oxyz, plus the Entrained Acceleration of the Rotating Oxyz, plus the Coriolis Term.



ROTATING FRAME OF REFERENCE $Oxyz$ TURNS WITH ANGULAR VELOCITY $\vec{\Omega}$.

BEGIN BY STATING THAT FOR ANY VECTOR $\vec{Q}$ IN A ROTATING FRAME OF REFERENCE $Oxyz$, THE TIME RATE OF CHANGE $\dot{\vec{Q}}$ IS:

$$\frac{d\vec{Q}}{dt} = \underbrace{\vec{\Omega}\times\vec{Q}}_{\text{ENTRAINED TERM}} + \underbrace{\left(\frac{d\vec{Q}}{dt}\right)_{Oxyz}}_{\text{RELATIVE TERM}}$$

NOW LET $\vec{Q} = \vec{r}$, THE DISPLACEMENT VECTOR.

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\Omega}\times\vec{r} + \left(\frac{d\vec{r}}{dt}\right)_{Oxyz}$$

DIFFERENTIATE ONCE MORE:

$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{\Omega}\times\left[\vec{\Omega}\times\vec{r} + \left(\frac{d\vec{r}}{dt}\right)_{Oxyz}\right] + \frac{d}{dt}\left\{\vec{\Omega}\times\vec{r} + \left(\frac{d\vec{r}}{dt}\right)_{Oxyz}\right\}\Bigg|_{Oxyz}$$

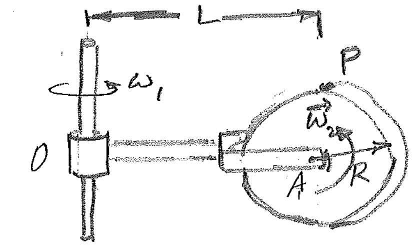
$$= \vec{\Omega}\times\vec{\Omega}\times\vec{r} + \vec{\Omega}\times\left(\frac{d\vec{r}}{dt}\right)_{Oxyz}$$

LIKE TERMS

$$\vec{\alpha} \longrightarrow + \left(\frac{d\vec{\Omega}}{dt}\right)_{Oxyz}\times\vec{r} + \vec{\Omega}\times\left(\frac{d\vec{r}}{dt}\right)_{Oxyz} + \left(\ddot{\vec{r}}\right)_{Oxyz}$$

$$= \underbrace{\vec{\Omega}\times\vec{\Omega}\times\vec{r} + \vec{\alpha}\times\vec{r}}_{\text{ENTRAINED TERMS}} + \underbrace{\left(\ddot{\vec{r}}\right)_{Oxyz}}_{\text{RELATIVE TERM}} + \underbrace{2\,\vec{\Omega}\times\left(\dot{\vec{r}}\right)_{Oxyz}}_{\text{CORIOLIS TERM}}$$

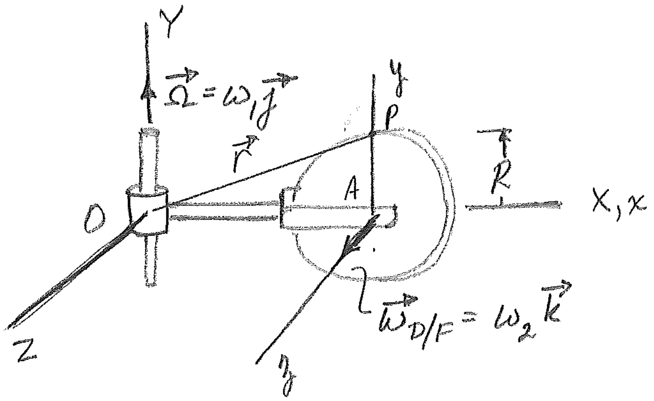
Example



The Disk shown rotates with a Constant angular acceleration ω_2, and the arm OA rotates with a constant rate ω_1.

Determine: (a) $\vec{v}_P$ (b) $\vec{a}_P$ (c) $\vec{\omega}_{DISK}$ (d) $\vec{\alpha}_{DISK}$

Frames of Reference   Fixed: OXYZ   Rotating Axyz



Velocity $\vec{v}_P$:

$$\vec{v}_P = \vec{v}_{ent.} + \vec{v}_{Rel}$$

$$\vec{v}_{ent} = \vec{\Omega} \times \vec{r} = (\omega_1 \vec{j}) \times (L\vec{i} + R\vec{j}) = -\omega_1 L \vec{k}$$

$$\vec{v}_{Rel} = \vec{\omega}_{D/F} \times \vec{r}_{P/A} = (\omega_2 \vec{k}) \times (R\vec{j}) = -\omega_2 R \vec{i}$$

$$\therefore \quad \vec{v}_P = -\omega_2 R \vec{i} - \omega_1 L \vec{k}$$

Acceleration of P:

$$\vec{a}_P = \vec{a}_{ent} + \vec{a}_{rel} + \vec{a}_{COR}$$

$$\vec{a}_{ent} = \overset{0}{\cancel{\vec{\alpha} \times \vec{r}}} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = (\omega_1 \vec{j}) \times (-\omega_1 L \vec{k}) = -\omega_1^2 L \vec{i}$$

$$\vec{a}_{rel} = \vec{\omega}_{D/F} \times (\vec{\omega}_{D/F} \times \vec{r}_{P/A}) = (\omega_2 \vec{k}) \times (-\omega_2 R \vec{i}) = -\omega_2^2 R \vec{j}$$

$$\vec{a}_{COR} = 2\vec{\Omega} \times \vec{v}_{Rel} = 2 \cdot (\omega_1 \vec{j}) \times (-\omega_2 R \vec{i}) = 2\omega_1 \omega_2 R \vec{k}$$

$$\therefore \quad \vec{a}_P = -\omega_1^2 L \vec{i} - \omega_2^2 R \vec{j} + 2\omega_1 \omega_2 R \vec{k}$$

Angular Velocity of Disk:

$$\vec{\omega}_{DISK} = \vec{\Omega} + \vec{\omega}_{D/F} \Rightarrow \vec{\omega}_{DISK} = \omega_1 \vec{j} + \omega_2 \vec{k}$$

Angular Acceleration of Disk:

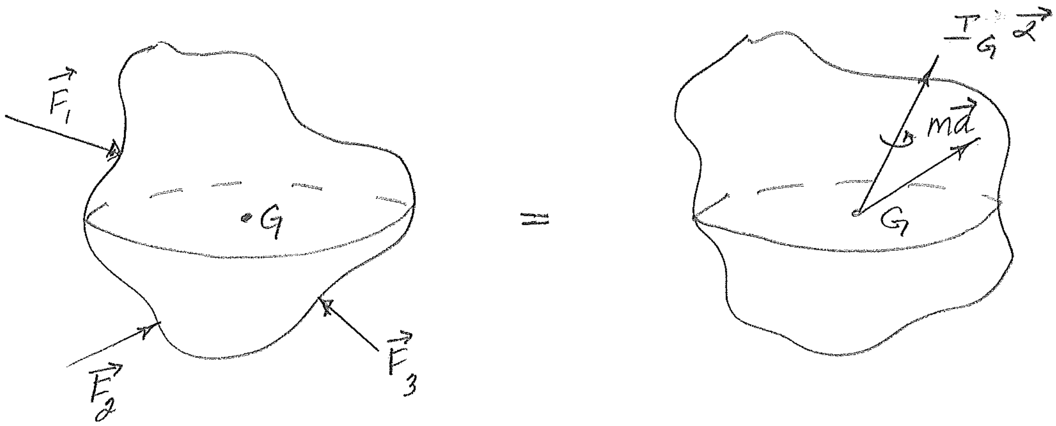
$$\vec{\alpha} = (\dot{\vec{\omega}})_{OXYZ} = (\dot{\vec{\omega}})_{Axyz} + \vec{\Omega} \times \vec{\omega}$$

$$= 0 + (\omega_1 \vec{j}) \times (\omega_1 \vec{j} + \omega_2 \vec{k})$$

$$\Rightarrow \vec{\alpha}_{DISK} = \omega_1 \omega_2 \vec{i}$$

2. Kinetics

When we considered kinetics of particles we equated the sum of all external forces with the effective force, $m\vec{a}$. For rigid body kinetics we must also consider the external moments as well.

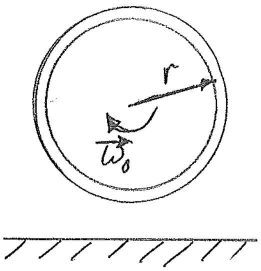


$$\sum F_{ext} = m\vec{a}$$

$$\left\{ \sum M_G \right\}_{ext} = I_G \vec{\alpha}$$

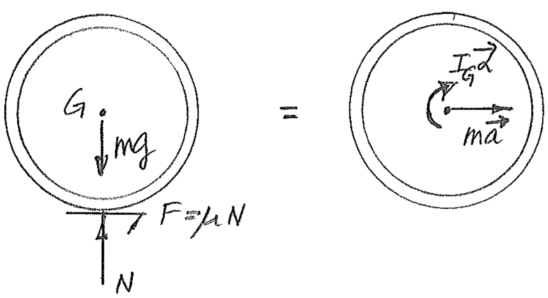
Kinetics of Rigid Bodies also requires extensive use of Free Body Diagrams.

Example



A thin hoop of mass m and radius r is dropped onto a horizontal surface with an initial angular velocity $\vec{\omega}_0$. If μ is the coefficient of friction between the hoop and floor, Determine (a) the time t_1 at which the hoop begins rolling without slipping, (b) the linear and angular velocities of the hoop at t_1

For the hoop, we have $I_G = mr^2$



$+\uparrow \sum F_y = 0: \quad N - mg = 0 \Rightarrow N = mg$

$\xrightarrow{+} \sum F_x = ma: \quad \mu mg = ma \Rightarrow a = \mu g$

$+\circlearrowright \sum M_G = I_G \alpha: \quad -\mu mgr = mr^2 \alpha \Rightarrow \alpha = \frac{\mu g}{r}$

Kinematics of Motion:

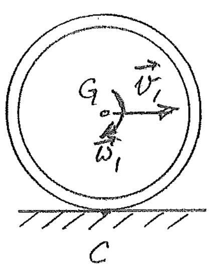
During the period of sliding, we have uniformly accelerated motion.

@ $t = 0$, $v_G = 0$: $\quad v_G = v_0 + a_G t$

$v_G = 0 + \mu g t$

@ $t = 0$, $\omega = \omega_0$: $\quad \omega = \omega_0 + \alpha t$

$\omega = \omega_0 - \frac{\mu g}{r} t$



The hoop will roll without slipping at $t = t_1$, when point C is the instantaneous center of rotation.

$$v_1 = r\omega_1$$

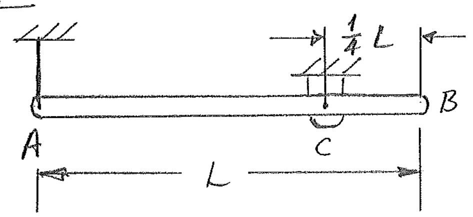
$$\mu g t_1 = r\left[\omega_0 - \left(\frac{\mu g}{r}\right) t_1\right] \Rightarrow t_1 = \frac{r\omega_0}{2\mu g}$$

Substitute t_1 into kinematic relations:

$$v_1 = \mu g \left[\frac{r\omega_0}{2\mu g}\right] = \frac{1}{2} r\omega_0 \Rightarrow \vec{v}_1 = \frac{1}{2} r\omega_0 \rightarrow$$

$$\omega_1 = \omega_0 - \frac{\mu g}{r}\left[\frac{r\omega_0}{2\mu g}\right] = \frac{1}{2}\omega_0 \Rightarrow \vec{\omega}_1 = \frac{1}{2}\omega_0$$

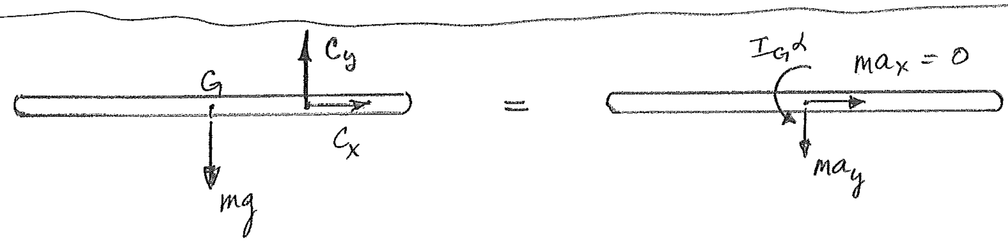
Example



Uniform beam supported as shown. If the cable at A suddenly breaks, determine

(a) The reaction at C

(b) The acceleration of B



For a slender uniform rod, $I_G = \frac{1}{12} mL^2$

$\xrightarrow{+} \sum F_x = ma_x$: $C_x = ma_x$ (1) $\Rightarrow C_x = 0$

$+\downarrow \sum F_y = ma_y$: $mg - C_y = ma_y$ (2)

$+\circlearrowleft \sum M_G = I_G\alpha$: $C_y \cdot \frac{L}{4} = \frac{1}{12} mL^2\alpha$ (3)

From kinematics : $a_y = \alpha \cdot \frac{L}{4}$ (4)

Subst. (4) into (2) : $mg - C_y = m \cdot \alpha \frac{L}{4}$

mult. by $\frac{L}{4}$: $mg \cdot \frac{L}{4} - C_y \frac{L}{4} = m \cdot \alpha \frac{L^2}{16}$

Add to (3) : $\frac{L}{4} mg = m\alpha \frac{L^2}{16} + m\alpha \frac{L^2}{12} \Rightarrow \alpha = \frac{12g}{7L}$ $\circlearrowleft$

Subst. into (2) solve for C_y $\Rightarrow C_y = \frac{4mg}{7}$

Acceleration of B : $a_B = \frac{L}{4}\alpha$

$= \frac{L}{4} \cdot \frac{12g}{7L}$

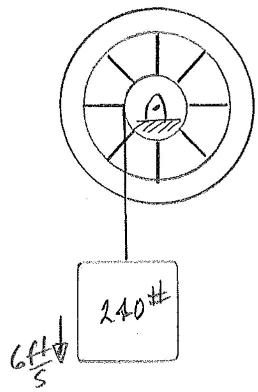
$a_B = \frac{3}{7} g \uparrow$

3. Work-Energy Principle

As with particles, $T_2 = T_1 + U_{1\to2}$

But for rigid bodies: $T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$

Example:



The 240# block is suspended from an inextensible cable wrapped around a drum of 1.25 ft radius. For the drum and flywheel, $I_G = 10.5$ lb·ft·s². For the instant shown, the block is moving downwards at 6 ft/s.

The bearing is poorly lubricated, and exerts a frictional couple of 60#' on the drum.

Determine the velocity of the block after it has moved through 4 ft.

Kinetic Energy: $T_1 = \frac{1}{2} m v^2 + \frac{1}{2} I_G \omega^2, \quad \omega_1 = \frac{6\ ft/s}{1.25\ ft} = 4.80\ \frac{rad}{s}$

$$T_1 = \frac{1}{2}\left(\frac{240\#}{32.2}\right)\left(6\ \frac{ft}{s}\right)^2 + \frac{1}{2}(10.5\ \#\cdot ft\cdot s^2)\left(4.80\ \frac{rad}{s}\right)^2$$

$$T_1 = 255\ \#'$$

Now Position 2: $T_2 = \frac{1}{2}\left(\frac{240}{32.2}\right) v_2^2 + \frac{1}{2}(10.5)\frac{v_2^2}{(1.25)^2} \Rightarrow T_2 = 7.09\, v_2^2$

Work: Only weight and friction couple do work.

$s_1 = 0 \qquad s_2 = 4\ ft$

$\theta_1 = 0 \qquad \theta_2 = \frac{4\ ft}{1.25\ ft} = 3.20\ rad.$

$$U_{1\to2} = W(s_2 - s_1) - M(\theta_2 - \theta_1)$$

$$= (240\#)(4-0) - (60\#')(3.2-0) \Rightarrow U_{1\to2} = 768\ \#'$$

Work-Energy Principle: $T_2 = T_1 + U_{1\to2}$

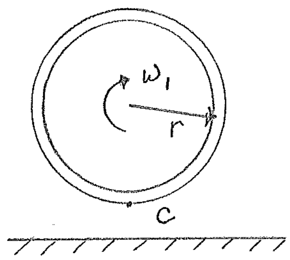
$$7.09\, v_2^2 = 255 + 768$$

$$v_2 = 12.01\ \frac{ft}{sec}$$

4. Principle of Impulse-Momentum

For rigid body I-M problems, the moments of the external impulses must be considered also.

Example:



Solve the example on page 6 by the principle of Impulse-Momentum

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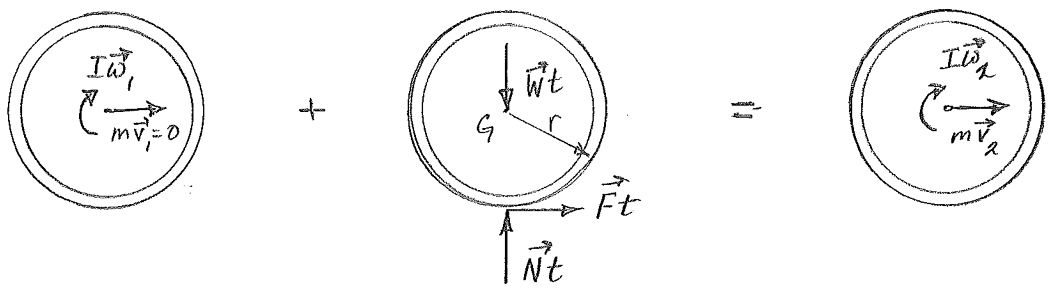
$I_G = mr^2$



$$\{\text{System Momenta}\}_1 + \{\text{System External Impulses}\}_{1\to 2} = \{\text{System Momenta}\}_2$$

$+\uparrow$ y components: $\quad 0 + Nt - Wt = 0 \quad (1) \Rightarrow \quad N = W = mg$

$\xrightarrow{+}$ x components: $\quad 0 + Ft = mv_2 \quad (2)$

$+\circlearrowright$ moments about G: $\quad -mr^2\omega_1 + (Ft)r = -mr^2\omega_2 \quad (3)$

From (1) and (2): $\quad \mu mgt = mv_2 \Rightarrow v_2 = \mu g t \quad (4)$

From (3): $\quad -mr^2\omega_1 + \mu mgtr = -mr^2\omega_2 \Rightarrow \omega_2 = \omega_1 - \frac{\mu g}{r}t \quad (5)$

For rolling w/o slipping $\quad v_2 = r\omega_2$

$$\mu g t_2 = r\left(\omega_1 - \frac{\mu g}{r}t_2\right) \Rightarrow t_2 = \frac{r\omega_1}{2\mu g}$$

Subst. into (4) & (5): $\quad v_2 = \mu g\left(\frac{r\omega_1}{2\mu g}\right) \Rightarrow v_2 = \frac{1}{2}r\omega_1 \rightarrow$

$$\omega_2 = \omega_1 - \frac{\mu g}{r}\cdot\frac{r\omega_1}{2\mu g} \Rightarrow \omega_2 = \frac{1}{2}\omega_1 \circlearrowright$$