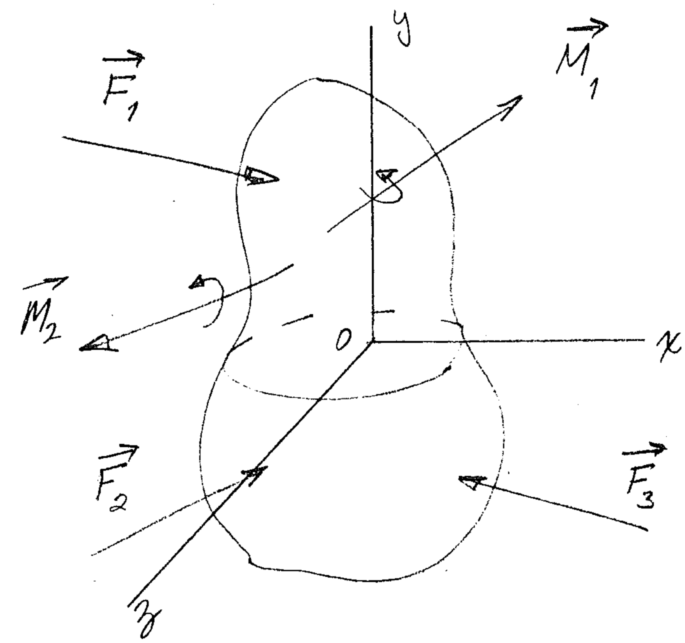
3-DIMENSIONAL EQUILIBRIUM



CONSIDER THE RIGID BODY AT LEFT UNDER THE INFLUENCE OF SEVERAL EXTERNAL FORCES AND MOMENTS, AS SHOWN. WHEN THE SYSTEM IS IN EQUILIBRIUM, WE MAY WRITE:

$$\sum \vec{F} = 0 \quad \text{AND} \quad \sum \vec{M}_0 = 0$$

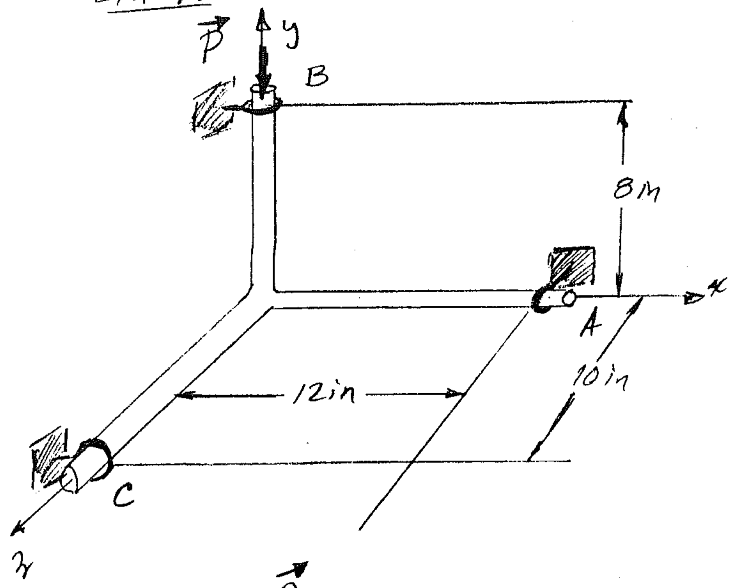
THESE TWO VECTOR EQUATIONS ARE EQUIVALENT TO SIX SCALAR EQUATIONS:

$$\sum F_x = 0 \qquad \sum M_x = 0$$

$$\sum F_y = 0 \qquad \sum M_y = 0$$

$$\sum F_z = 0 \qquad \sum M_z = 0$$

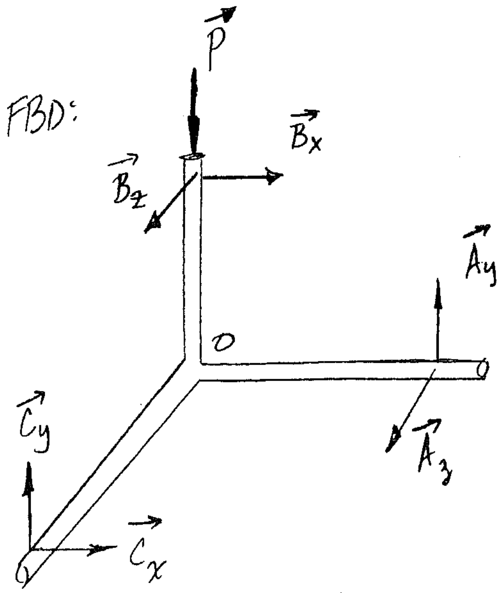
EXAMPLE:



KNOWING THAT $\vec{P}$ IS 250 #, DETERMINE THE REACTIONS AT A, B AND C.

A, B, AND C ARE "SMOOTH" EYEBOLTS.

FBD:



$$\Sigma \vec{F} = 0$$

$$A_y\vec{j} + A_z\vec{k} + B_x\vec{i} + B_z\vec{k} + C_x\vec{i} + C_y\vec{j} - \vec{P} = 0$$

$X: \quad B_x + C_x = 0 \quad \Rightarrow \quad C_x = -B_x \quad (1)$

$Y: \quad A_y + C_y - P = 0 \quad \Rightarrow \quad C_y = 250 - A_y \quad (2)$

$Z: \quad A_z + B_z = 0 \quad \Rightarrow \quad B_z = -A_z \quad (3)$

$$\Sigma \vec{M}_C = \Sigma(\vec{r} \times \vec{F}) = 0$$

$$(12\vec{i} - 10\vec{k}) \times (A_y\vec{j} + A_z\vec{k}) + (8\vec{j} - 10\vec{k}) \times (B_x\vec{i} + B_z\vec{k}) + (8\vec{j} - 10\vec{k}) \times (-250\vec{j}) = 0$$

$$12A_y\vec{k} - 12A_z\vec{j} + 10A_y\vec{i} - 8B_x\vec{k} + 8B_z\vec{i} - 10B_x\vec{j} - 2500\vec{i} = 0$$

$X: \quad 10A_y + 8B_z - 2500 = 0 \quad \Rightarrow \quad 10A_y + 8B_z = 2500 \quad (4)$

$Y: \quad -12A_z - 10B_x = 0 \quad \Rightarrow \quad B_x = -\frac{6}{5}A_z \quad (5)$

$Z: \quad 12A_y - 8B_x = 0 \quad \Rightarrow \quad B_x = \frac{3}{2}A_y \quad (6)$

COMBINE (5) and (6) $\quad -\frac{6}{5} A_z = \frac{3}{2} A_y \Rightarrow A_y = -\frac{4}{5} A_z$

SUBST. INTO (4) w/ (3) $\quad 10\left(-\frac{4}{5} A_z\right) + 8\left(-A_z\right) = 2500$

$$-16 A_z = 2500 \Rightarrow A_z = -156\tfrac{1}{4}\#$$

WITH A_z NOW KNOWN, $\quad A_y = -\frac{4}{5}\left(-156\tfrac{1}{4}\right) \Rightarrow A_y = {}^{+}125\#$

$$B_z = -\left(-156\tfrac{1}{4}\right) \Rightarrow B_z = {}^{+}156\tfrac{1}{4}\#$$

REWRITE (2) AS: $\quad C_y = 250 - (125) \Rightarrow C_y = {}^{+}125\#$

REWRITE (6) AS: $\quad B_x = \frac{3}{2}(125) \Rightarrow B_x = {}^{+}187\tfrac{1}{2}\#$

REWRITE (1) AS: $\quad C_x = -\left(187\tfrac{1}{2}\right) \Rightarrow C_x = {}^{-}187\tfrac{1}{2}\#$

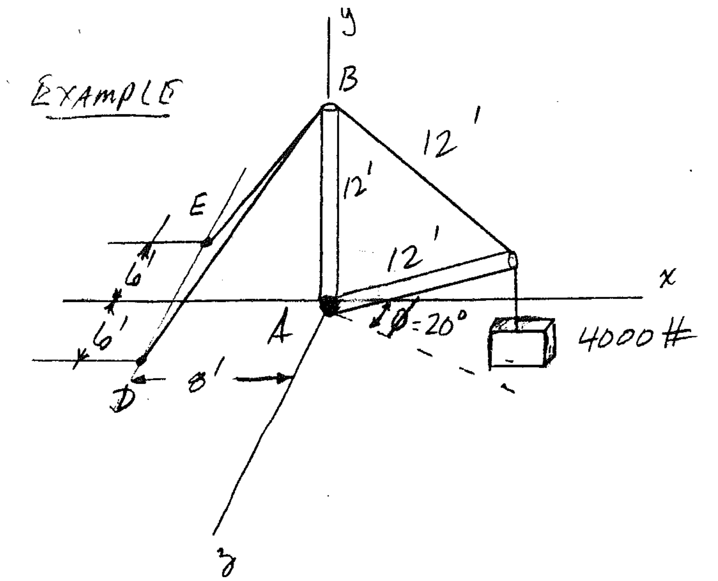
REACTIONS @ A, B and C ARE:

$$\vec{A} = (125\#)\vec{j} - (156\tfrac{1}{4}\#)\vec{k}$$

$$\vec{B} = (187\tfrac{1}{2}\#)\vec{i} + (156\tfrac{1}{4}\#)\vec{k}$$

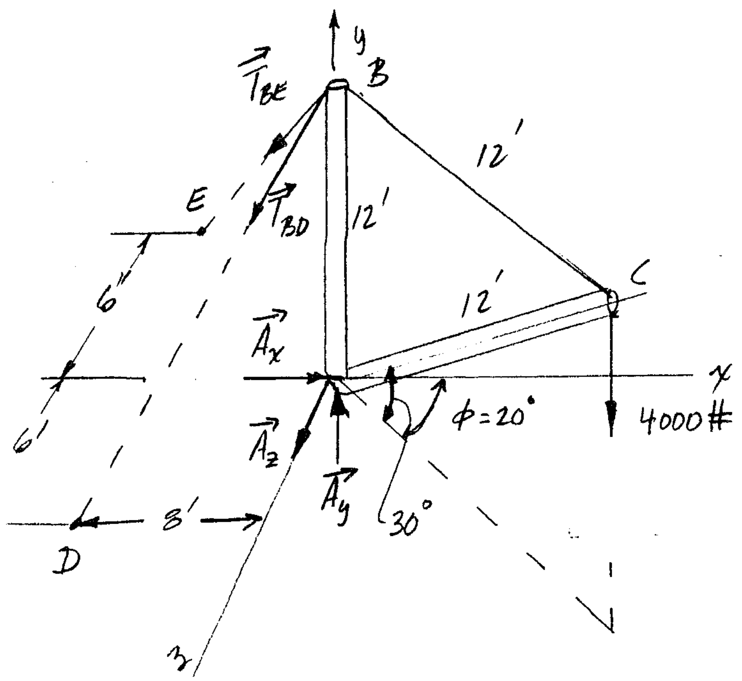
$$\vec{C} = (-187\tfrac{1}{2}\#)\vec{i} + (156\tfrac{1}{4}\#)\vec{k}$$

EXAMPLE



DETERMINE THE TENSION IN EACH CABLE AND THE REACTION @ A.

FREE-BODY DIAGRAM:



$$\vec{BD} = -8\vec{i} - 12\vec{j} + 6\vec{k} \text{ ft}, \quad BD = 15.62'$$

$$\vec{BE} = -8\vec{i} - 12\vec{j} - 6\vec{k} \text{ ft}, \quad BE = 15.62'$$

$$\vec{T}_{BD} = \frac{T_{BD}}{BD} \cdot \vec{BD}$$

$$\vec{T}_{BD} = T_{BD}\left(-0.512\vec{i} - 0.768\vec{j} + 0.348\vec{k}\right)$$

$$\vec{T}_{BE} = \frac{T_{BE}}{BE} \cdot \vec{BE}$$

$$\vec{T}_{BE} = T_{BE}\left(-0.512\vec{i} - 0.768\vec{j} - 0.348\vec{k}\right)$$

$$\Sigma\vec{F} = 0: \quad A_x\vec{i} + A_y\vec{j} + A_z\vec{k} + \vec{T}_{BD} + \vec{T}_{BE} - (4000\#)\vec{j} = 0$$

X: $A_x - 0.512\,T_{BD} - 0.512\,T_{BE} = 0$ (1)

Y: $A_y - 0.768\,T_{BD} - 0.768\,T_{BE} - 4000 = 0$ (2)

Z: $A_z + 0.348\,T_{BD} - 0.348\,T_{BE} = 0$ (3)

CONTINUED,

$$\vec{AC} = 12'\left(\cos 30^\circ \cos 20^\circ \vec{i} + \sin 30^\circ \vec{j} + \cos 30^\circ \sin 20^\circ \vec{k} \right)$$

$$\Sigma \vec{M}_A = \Sigma(\vec{r} \times F) = 0: \qquad \vec{AC} \times \vec{W} + \vec{AB} \times \vec{T}_{BD} + \vec{AB} \times \vec{T}_{BE} = 0$$

$$\vec{AC} \times \vec{W} = \left[(9.77\vec{i} + 6\vec{j} + 3.55\vec{k})\ ft \right] \times \left[(-4000\vec{j})\ \# \right]$$

$$= -39{,}062\vec{k} + 14{,}218\vec{i}\ \#'$$

$$\vec{AB} \times \vec{T}_{BD} = \left[(12\vec{j})\ ft \right] \times \left[T_{BD}(-0.512\vec{i} - 0.768\vec{j} + 0.348\vec{k})\ \# \right]$$

$$= 6.146\, T_{BD}\, \vec{k} + 4.609\, T_{BD}\, \vec{i}\ \#'$$

$$\vec{AB} \times \vec{T}_{BE} = \left[(12\vec{j})\ ft \right] \times \left[T_{BE}(-0.512\vec{i} - 0.768\vec{j} - 0.348\vec{k})\ \# \right]$$

$$= 6.146\, T_{BE}\, \vec{k} - 4.609\, T_{BE}\, \vec{i}\ \#'$$

$$\left.\begin{array}{lll} X: & 14{,}218 + 4.609\, T_{BD} - 4.609\, T_{BE} = 0 & (4) \\ Z: & -39{,}062 + 6.146\, T_{BD} + 6.146\, T_{BE} = 0 & (5) \end{array}\right\} \text{ SOLVE FOR } T_{BE} \,\&\, T_{BD}$$

$$\Rightarrow \quad T_{BD} = 1640\# \qquad T_{BE} = 4720\#$$

Subst. into (1): $A_x = 0.512(1640) + 0.512(4720) \Rightarrow A_x = {}^{+}3260\#$

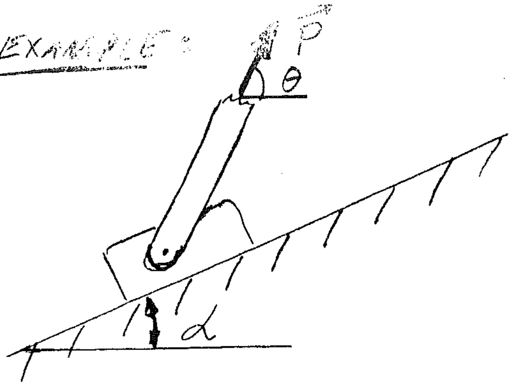
" " (2): $A_y = 0.768(1640) + 0.768(4720) + 4000 \Rightarrow A_y = {}^{+}8880\#$

" " (3): $A_z = 0.348(4720) - 0.348(1640) \Rightarrow A_z = {}^{+}1183\#$

REACTION MAGNITUDE: $A = \sqrt{(3260)^2 + (8880)^2 + (1183)^2}$

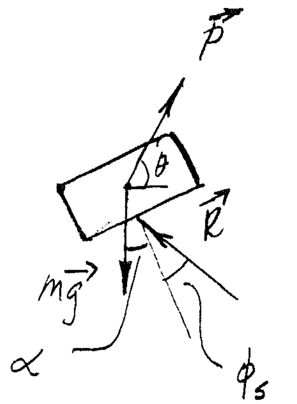
$A = 9533\#$

EXAMPLE:



LET ϕ_s DENOTE THE STATIC ANGLE OF FRICTION. DETERMINE THE MAGNITUDE AND DIRECTION OF $\vec{P}$ THAT WILL CAUSE THE BLOCK TO MOVE UP THE INCLINED PLANE

FREE BODY DIAGRAM:



$$\xrightarrow{+} \Sigma F_x = 0: \quad P\cos\theta - R\sin(\alpha + \phi_s) = 0$$

$$R = \frac{P\cos\theta}{\sin(\alpha+\phi_s)}$$

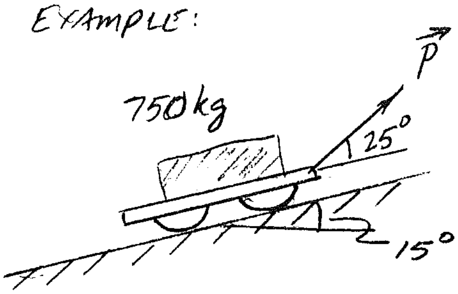
$$+\uparrow \Sigma F_y = 0: \quad P\sin\theta - mg + R\cos(\phi_s + \alpha) = 0$$

$$P\sin\theta - mg + \frac{P\cos\theta}{\sin(\alpha+\phi_s)} \cdot \cos(\alpha+\phi_s) = 0$$

$$P\left[\sin\theta + \cos\theta \cdot \cot(\alpha+\phi_s)\right] = mg$$

$$P = \frac{mg}{\sin\theta + \cos\theta\cot(\alpha+\phi_s)}$$

EXAMPLE:



GIVEN THAT THE MASS OF THE BLOCK & SLED IS 750 kg AND THAT $\mu_s = 0.40$ and $\mu_k = 0.30$ DETERMINE

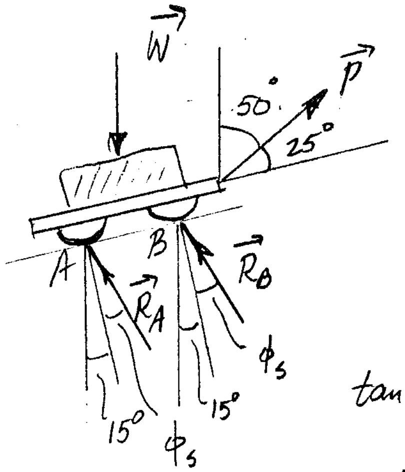
(a) THE FORCE $\vec{P}$ TO START THE SLED MOVING

(b) THE FORCE $\vec{P}$ TO KEEP THE SLED MOVING AFTER IT HAD STARTED MOVING

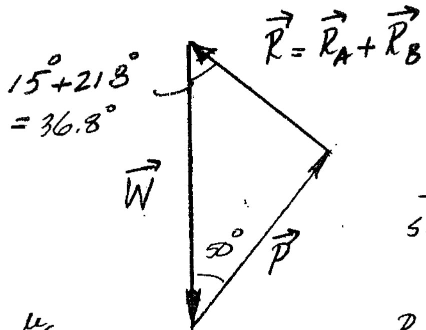
(c) THE FORCE $\vec{P}$ TO KEEP THE SLED FROM SLIDING DOWN THE INCLINE

FREE-BODY DIAGRAMS:

(a)



$\tan\phi_s = \mu_s$
$= 0.40$
$\phi_s = 21.8°$



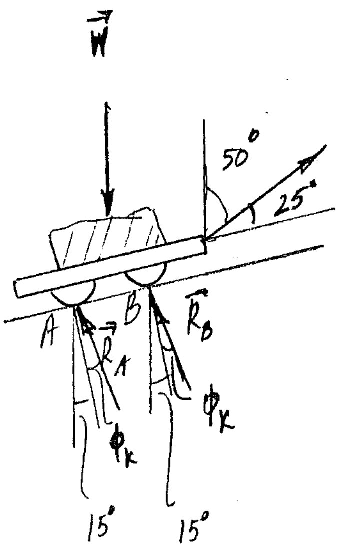
$$\frac{P}{\sin 36.8°} = \frac{W}{\sin(180 - 50 - 36.8)}$$

$$P = \frac{\sin(36.8)}{\sin(93.2°)} \cdot (750)(9.81)$$

$P = 4414\ N$

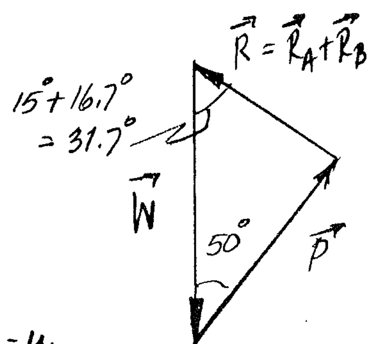
$P = 4.41\ kN$

(b)



$\tan\phi_k = \mu_k$
$= 0.30$

$\phi_k = 16.7°$



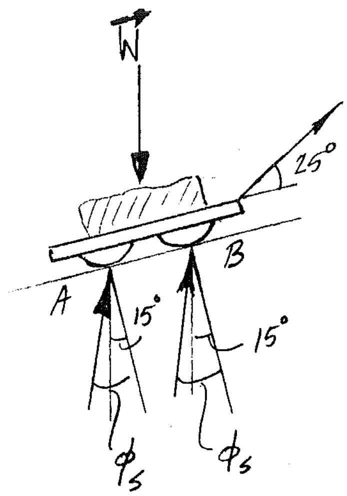
$$\frac{P}{\sin(31.7°)} = \frac{W}{\sin(180 - 50 - 31.7)}$$

$$P = \frac{\sin(31.7)}{\sin(} (750\,kg)(9.81\,\tfrac{m}{s^2})$$

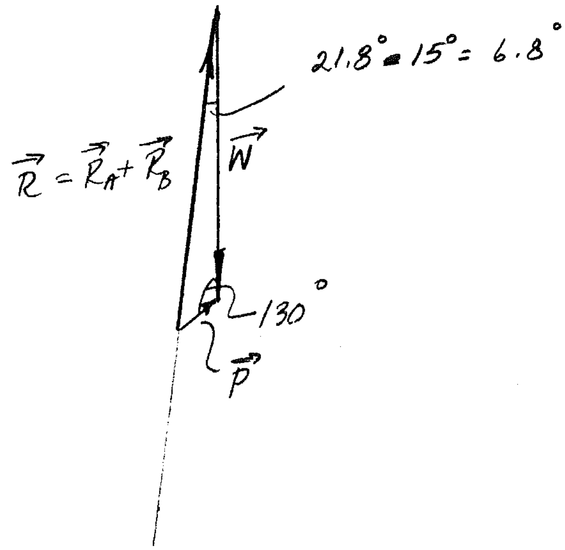
$P = 3907\ N$

$P = 3.91\ kN$

(c)



$\phi_s = 21.8°$



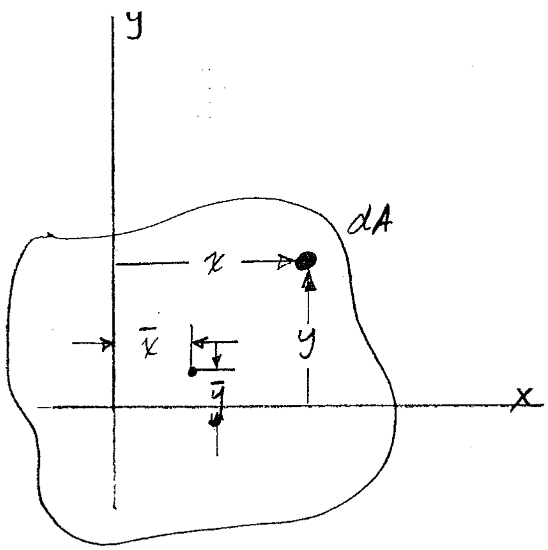
$$\frac{P}{\sin(6.8°)} = \frac{W}{\sin(180 - 130° - 6.8)}$$

$$P = \frac{\sin(6.8°)}{\sin(43.2°)} (750\,\text{kg})(9.81\,\tfrac{m}{s^2})$$

$$= 1273\ N$$

$$P = 1.273\ kN$$

CENTROID OF AREA



TO DETERMINE THE CENTROID OF AREA, CONSIDER THE MOMENT OF THE AREA ELEMENT dA ABOUT EACH AXIS.

$$dM_x = y\,dA$$

$$dM_y = x\,dA$$

INTEGRATE OVER THE AREA –

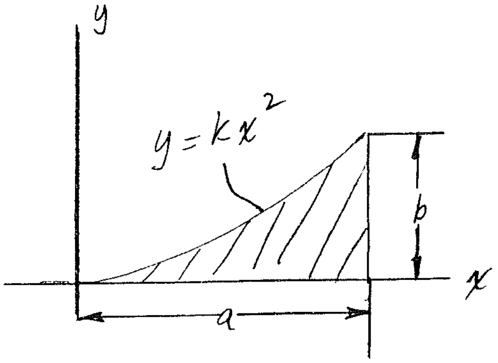
$$\left.\begin{aligned} M_x &= \int dM_x = \int y\,dA \\ M_y &= \int dM_y = \int x\,dA \end{aligned}\right\}$$

CALLED THE FIRST MOMENT OF AREA

THE CENTROID OF AREA IS GIVEN BY:

$$\bar{x}A = \int x\,dA$$

$$\bar{y}A = \int y\,dA$$



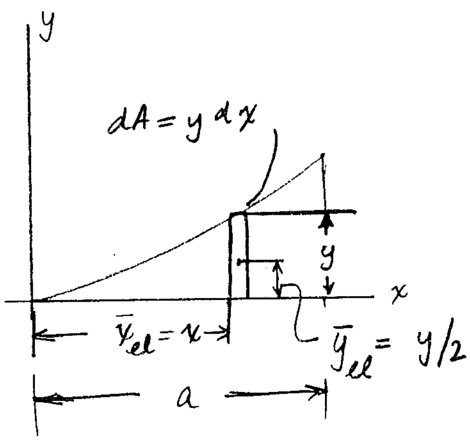
DETERMINE BY DIRECT INTEGRATION THE CENTROID OF AREA OF THE PARABOLIC SPANDREL.

FIRST DETERMINE k :

$$b = k \cdot (a)^2 \implies k = \frac{b}{a^2}$$

$$y = \frac{b}{a^2} \cdot x^2 \quad \text{OR} \quad x = \frac{a}{b^{1/2}} \cdot y^{1/2}$$

VERTICAL ELEMENTS:



$$A = \int dA = \int y\,dx = \int_0^a \frac{b}{a^2} x^2 \cdot dx$$

$$= \frac{b}{a^2}\left[\frac{1}{3}x^3\right]\Big|_0^a$$

$$A = \frac{ab}{3}$$

$$\int \bar{x}_{el}\,dA = \int xy\,dx = \int_0^a x\left(\frac{b}{a^2}x^2\right)dx = \frac{b}{a^2}\left[\frac{1}{4}x^4\right]\Big|_0^a$$

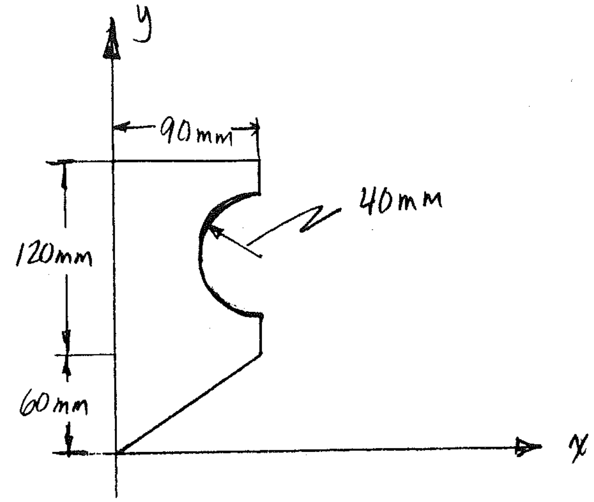
$$= \frac{a^2 b}{4}$$

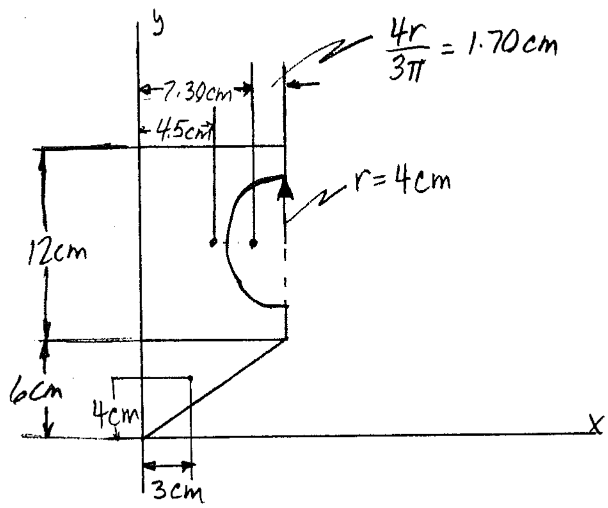
$$\bar{x} = \frac{\int \bar{x}_{el}\,dA}{A} = \frac{a^2b/4}{ab/3} \implies \bar{x} = \frac{3}{4}a$$

$$\int \bar{y}_{el}\,dA = \int \frac{y}{2} \cdot y\,dx = \int_0^a \frac{1}{2}\left(\frac{b}{a^2}x^2\right)^2 dx = \frac{1}{2} \cdot \frac{b^2}{a^4}\left[\frac{1}{5}x^5\right]\Big|_0^a = \frac{ab^2}{10}$$

$$\bar{y} = \frac{\int \bar{y}_{el}\,dA}{A} = \frac{ab^2/10}{ab/3} \implies \bar{y} = \frac{3}{10}b$$

DETERMINE THE CENTROID OF AREA





COMPONENT	AREA	$\bar{x}$ (cm)	$\bar{y}$ (cm)	$\bar{x}A$ (cm^3)	$\bar{y}A$ (cm
RECTANGLE	+108	4.50	12	+486	+1296
TRIANGLE	+27	3.00	4	+81	+108
SEMICIRCLE	-25.1	7.30	12	-183	-301

Sum OF AREAS : $\Sigma A = 109.9 \text{ cm}^2$

Sum of $\bar{x}A$'s : $\Sigma \bar{x}A = 384 \text{ cm}^3$

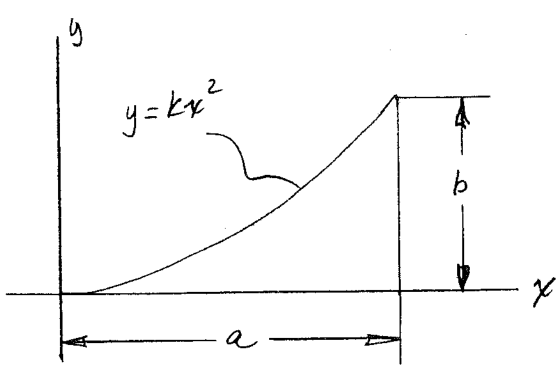
SUM OF $\bar{y}A$'s : $\Sigma \bar{y}A = 1103 \text{ cm}^3$

CENTROID COORDINATES : $\bar{X}\,\Sigma A = \Sigma \bar{x} A$

$$\bar{X} = \frac{\Sigma \bar{x}A}{\Sigma A} = \frac{384 \text{ cm}^3}{109.9 \text{ cm}^2} \Rightarrow \bar{X} = 3.49 \text{ cm}$$

$$\bar{Y}\,\Sigma A = \Sigma \bar{y}A$$

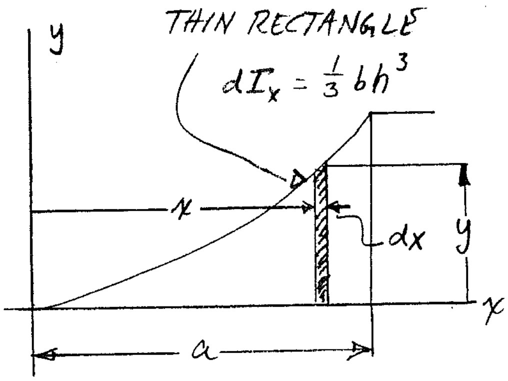
$$\bar{Y} = \frac{\Sigma \bar{y}A}{\Sigma A} = \frac{1103 \text{ cm}^3}{109.9 \text{ cm}^2} \Rightarrow \bar{Y} = 10.04 \text{ cm}$$



(a) DETERMINE I_x and I_y

(b) USING THESE RESULTS DETERMINE THE RADII OF GYRATION k_x ; k_y

(a) RECALL, ... $\quad k = \dfrac{b}{a^2} \quad\quad A = \frac{1}{3}ab$



MOMENT OF INERTIA I_x :

$$dI_x = \frac{1}{3} y^3 dx$$

$$= \frac{1}{3}\left(\frac{b}{a^2}x^2\right)^3 dx$$

$$= \frac{1}{3}\frac{b^3}{a^6} x^6 dx$$

$$I_x = \int_0^a \frac{1}{3}\frac{b^3}{a^6} x^6 dx = \frac{1}{3}\frac{b^3}{a^6}\left[\frac{1}{7}x^7\right]\Big|_0^a$$

$$\Rightarrow I_x = \frac{ab^3}{21}$$

MOMENT OF INERTIA I_y :

$$dI_y = x^2 dA = x^2 (y\,dx) = \frac{b}{a^2}x^4 dx$$

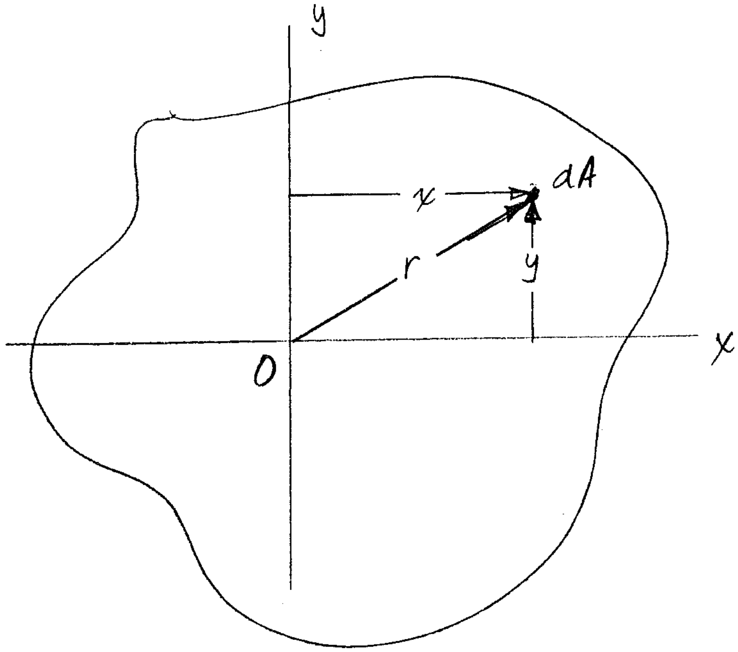
$$I_y = \int_0^a \frac{b}{a^2}x^4 dx = \frac{b}{a^2}\left[\frac{1}{5}x^5\right]\Big|_0^a = \frac{ba^3}{5} \Rightarrow I_y = \frac{ba^3}{5}$$

(b) RADII OF GYRATION :

$$k_x^2 = \frac{I_x}{A} = \frac{ab^3/21}{ab/3} \Rightarrow k_x = \sqrt{\frac{1}{7}}\,b$$

$$k_y^2 = \frac{I_y}{A} = \frac{ba^3/5}{ab/3} \Rightarrow k_y = \sqrt{\frac{3}{5}}\,a$$

MOMENTS OF INERTIA



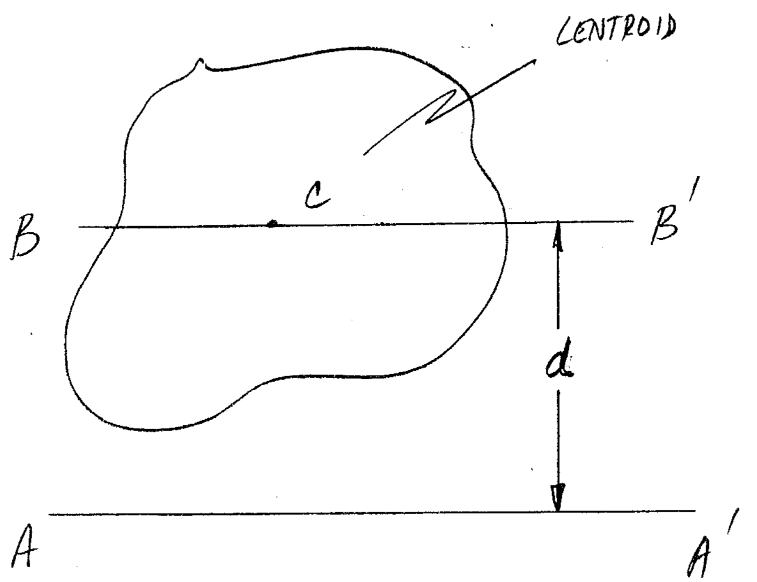
$$I_x = \int y^2 \, dA$$

$$I_y = \int x^2 \, dA$$

$$J_o = \int r^2 \, dA$$

Note that $r^2 = x^2 + y^2 \Rightarrow J_o = I_x + I_y$

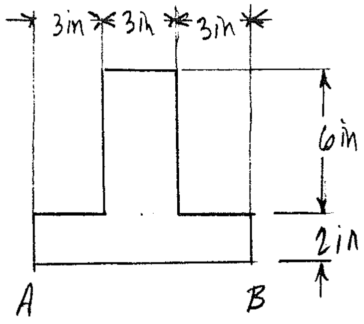
PARALLEL AXIS THEOREM:



LET $\bar{I}$ DENOTE THE MOMENT OF INERTIA ABOUT CENTROIDAL AXIS BB'. THE PARALLEL AXIS THEOREM STATES:

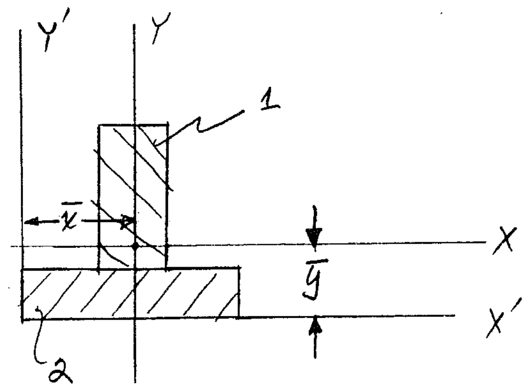
$$I = \bar{I} + Ad^2$$

MOMENT OF INERTIA ABOUT AXIS AA'



DETERMINE I_x AND I_Y WITH RESPECT TO CENTROIDAL AXES PARALLEL AND PERPENDICULAR TO SIDE AB.

CENTROID LOCATION: DEFINE $X'Y'$ AXES AS SHOWN



COMPONENT	AREA (in^2)	$\bar{x}$ (in)	$\bar{y}$ (in)	$\bar{x}A$ (in^3)	$\bar{y}A$ (in^3)
1	18	4.5	5	81	90
2	18	4.5	1	81	18
TOTAL	36	9	6	162	108

$$\bar{x}A = \Sigma \bar{x}A \Rightarrow \bar{x} = \frac{\Sigma \bar{x}A}{A} = \frac{162}{36} = 4.5 \Rightarrow \bar{x} = 4.5\,in$$

$$\bar{y}A = \Sigma \bar{y}A \Rightarrow \bar{y} = \frac{\Sigma \bar{y}A}{A} = \frac{108}{36} = 3.0 \Rightarrow \bar{y} = 3.0\,in$$

MOMENTS OF INERTIA:

$$I_x = \tfrac{1}{12}bh^3 + Ad^2 \qquad I_Y = \tfrac{1}{12}b^3h + Ad^2$$

RECTANGLE 1 $\quad \frac{1}{12}(3)(6)^3 + (18)(2)^2 = 126\,in^4 \quad \frac{1}{12}(3)^3(6) + \cancel{(18)(0)}^{\,0} = 13.5$

RECTANGLE 2 $\quad \frac{1}{12}(9)(2)^3 + (18)(2)^2 = 78\,in^4 \quad \frac{1}{12}(9)^3(2) + \cancel{(18)(0)}^{\,0} = 121.5$

COMPOSITE: $\quad I_x = 204\,in^4 \qquad I_Y = 135\,in^4$