Low-Flow Frequency Analysis Using Probability-Plot Correlation Coefficients

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Abstract: Although a vast amount of literature exists on the selection of an appropriate probability distribution for annual maximum floodflows, few studies have examined which probability distributions are most suitable to fit to sequences of annual minimum streamflows. Probability plots have been used widely in hydrology as a graphical aid to assess the goodness of fit of alternative distributions. Recently, probability-plot correlation-coefficient (PPCC) tests were introduced to test the normal, two-parameter lognormal and Gumbel hypotheses. Those procedures are extended here to include both regional and at-site tests for the two-parameter Weibull and lognormal distributional hypotheses. In theory, PPCC-hypothesis tests can only be developed for two-parameter distributions that exhibit a fixed shape. Nevertheless, the PPCC is a useful goodness-of-fit statistic for comparing three-parameter distributions. The PPCC derived from fitting the two- and three-parameter lognormal, two- and three-parameter Weibull, and log-Pearson type III distributions to sequences of annual minimum seven-day low flows at 23 sites in Massachusetts are compared. How the PPCC can be used to discriminate among both competing distributional hypotheses for the distributions of fixed shape and competing parameter-estimation procedures for the distributions with variable shape is described. An approximate regional PPCC test is developed and used to show that there is almost no evidence to contradict the hypothesis that annual minimum seven-day low flows in Massachusetts are two-parameter lognormal.

Introduction

With increasing attention focused on surface-water-quality management, many agencies routinely require estimates of the *d*-day, *T*-year, low flow for the maintenance of water-quality standards. Such low-flow statistics are now in common use for determining waste-load allocations, issuing and/or renewing National Pollution Discharge Elimination System (NPDES) permits, siting waste-treatment plants and sanitary landfills, and determining minimum downstream-release requirements from hydropower, irrigation, water-supply, and cooling-plant facilities. The most widely used index of low flow in the United States is the seven-day, ten-year low flow ($Q_{7,10}$), defined as the annual minimum average seven-day low flow that recurs, on average, once every ten years (Riggs et al. 1980). Estimation of the $Q_{7,10}$ from streamflow records consists of determination of a probability distribution of the annual minimum seven-day low flows and selection of a statistically efficient parameter-estimation procedure. Statisticians term these tasks distributional hypothesis testing and point estimation, respectively. If possible, these two tasks should be considered independently as is common practice in the field of applied statistics (Benjamin and Cornell 1970).

An extensive amount of literature exists on the selection of statistically efficient (minimum variance and unbiased) parameter-estimation procedures;

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in hydrology, most of those studies relate to fitting floodflow distributions. To our knowledge, only a few studies (Gumbel 1954, 1958; Matalas 1963; Condie and Nix 1975; Loganathan et al. 1985, 1986; Tasker 1987) have examined different parameter-estimation procedures for fitting alternative models to sequences of annual minimum seven-day low flows. Using four plausible three-parameter probability distributions, Condie and Nix (1975) compared alternative parameter-estimation procedures for their ability to generate acceptable lower bounds. According to Condie and Nix, an acceptable lower bound must fall in the interval between zero and the minimum observed flow. On the basis of this criterion, Condie and Nix recommend fitting the three-parameter Weibull (W3) distribution by the method of maximum likelihood (W3-MLE) if feasible solutions exist; otherwise they suggest using the method of smallest obsereved drought (W3-MSOD) or the method of moments (W3-MM), in that order. Tasker (1987) compared alternative three-parameter probability distributions and fitting procedures by the bootstrap method and recommended the use of either the log-Pearson type III (LP3) using method of moments (LP3-MM) or the Condie and Nix (1975) algorithm which was a close competitor. Matalas (1963) recommended the use of the W3 distribution fit using the method of moments (W3-MM) or the Pearson type III (P3) using maximum likelihood estimates (P3-MLE). Matalas did not consider the W3-MLE procedure, nor did he consider the LP3 distribution. These three important studies by Condie and Nix, Tasker, and Matalas were based on fitting alternative models to samples of annual minimum seven-day low flows at 38 gaging stations in Canada, 20 gaging stations in Virginia, and 14 gaging stations in the eastern United States, respectively. Nevertheless, their conclusions are consistent, considering the heterogeneity of the selected sites. Perhaps consistently good performance should not be surprising for the W3 distribution given that it is theoretically the parent model of extreme low flows (Gumbel 1954) and for the LP3 distribution given its extreme flexibility as evidenced in numerous studies of the distribution of annual peak floodflows. However, no cited studies have examined the adequacy of two-parameter distributions, such as the two-parameter lognormal (LN2) or the two-parameter Weibull (W2). The studies by Matalas (1963) and Condie and Nix (1975) rejected the three-parameter lognormal (LN3) model because they used the method of moments (LN3-MM). Stedinger (1980) introduced an improved fitting procedure for the LN3 distribution that would likely modify the conclusions of Condie and Nix and Matalas. Tasker (1987) did not include the LN3 distribution in his study because Condie and Nix and Matalas rejected that distribution.

Prior studies have shown that W3-MLE and LP3-MM provide two reasonable approaches to fitting low-flow frequency distributions, yet W3-MLE requires a relatively complex numerical algorithm not suitable for many practical applications and the LP3-MM procedure requires at-site estimates of the skew coefficient, which are not very precise for the small samples typically encountered (Wallis et al. 1974). When a two-parameter distribution provides an adequate description of annual minimum seven-day low flows, one need not estimate the skew coefficient; quantile estimators based upon a two-parameter distribution often have lower variance then three-parameter alternatives.

Recently, probability-plot correlation-coefficient (PPCC) tests have been introduced by Filliben (1975), Looney and Gulledge (1985), and Vogel (1986) for testing the normal and LN2 hypotheses and by Vogel (1986) for testing the Gumbel hypothesis. Such tests may be extended to other distributions that exhibit fixed shape, such as the uniform, exponential, and W2 hypotheses. This study develops new PPCC tests for the uniform and W2 hypotheses and uses those tests, in addition to the LN2 PPCC test, to evaluate the regional hypothesis that annual minimum seven-day low flows at 23 sites in Massachusetts arise from the two-parameter alternatives, LN2 and W2.

Although the PPCC test statistic cannot be formally used to test three-parameter distributional hypotheses, it can be used as a goodness-of-fit statistic that summarizes the linearity of a probability plot. Such procedures were originally suggested by Filliben (1975). Matalas (1963) used probability plots in his comparisons of the fit of W3 and Pearson Type III distributions to low flows.

In this paper, the PPCC is used to rank the goodness of fit of various parent probability models and to evaluate alternative parameter estimation procedures. In addition, PPCC-hypothesis tests are used to test various distributional hypotheses for low flows on a site-by-site and regional basis *without* reference to particular parameter-estimation procedures. These results make it evident that procedures for describing the distribution of annual minimum seven-day low flows are not limited to the W3-MLE and LP3-MM procedures in Massachusetts.

PROBABILITY-PLOT CORRELATION-COEFFICIENT HYPOTHESIS TESTS

Probability plots are used widely in the statistics and water resources literature. Although analytic approaches, such as the method of moments (MM) or maximum likelihood estimates (MLE) for fitting probability distributions to observed data, are, in theory, more efficient statistical procedures than graphical curve-fitting procedures, many practitioners would not make engineering decisions without the use of a graphical display (probability plot). Filliben (1975), Looney and Gulledge (1985), Vogel (1986), and others have proposed goodness-of-fit tests that are based upon information contained in probability plots. In this section, we review existing tests for the normal and lognormal hypotheses and introduce two new PPCC tests that will be used later to evaluate alternative frequency models for sequences of annual minimum seven-day low flows.

A probability plot is defined as a graphical representation of the *i*th-order statistic of the sample, $y_{(i)}$, as a function of a plotting position, which is simply a measure of the nonexceedance probability associated with the *i*th-order statistic from the assumed standardized distribution. The *i*th-order statistic is obtained by ranking the observed sample from the smallest ($i = 1$) to the largest ($i = n$) value, then $y_{(i)}$ equals the *i*th largest value. Many studies [for example, Cunnane (1978) and Arnell et al. (1986)] have recommended the use of unbiased plotting positions. Unbiased plotting positions reproduce the expected value of the *i*th-order statistic, $E[y_{(i)}]$ based on an assumed distribution. Although the original PPCC tests advocated by Filliben (1975) used a biased plotting position which reproduced the median value of the *i*th-order statistic, Looney and Gulledge (1985) showed that use of an unbiased plotting position for the normal distribution can lead to a more powerful (lower type II errors) hypothesis test. In this study, unbiased plotting positions are used, when available.

If the sample to be tested is actually distributed as hypothesized, one would expect the plot of the ordered observations, $y_{(i)}$, as a function of the order statistic means (M_i) to be approximately linear. The PPCC is simply a measure of the linearity of the probability plot. The PPCC test statistic is defined as the product moment correlation coefficient between the ordered observations and the order statistic means for each assumed distribution function. The PPCC test statistic is:

$$\hat{r} = \frac{\sum_{i=1}^{n} (y_{(i)} - \bar{y})(M_i - \bar{M})}{\sqrt{\sum_{i=1}^{n} (y_{(i)} - \bar{y})^2 \sum_{j=1}^{n} (M_j - \bar{M})^2}} \quad \text{(1)}$$

in which $M_i = E[y_{(i)}]$ for each hypothesized distribution; and $y_{(i)}$ = the ith largest observed value of the y_i. In general, to analytically construct a probability plot to estimate $\hat{r}$ in Eq. 1, the inverse of the hypothesized cumulative distribution must be obtained because

$$M_i = F_y^{-1}\{\hat{F}_y[y_{(i)}]\} \quad \text{(2)}$$

in which $\hat{F}_y(y_i)$ = an approximation to the nonexceedance probability associated with the ith-order statistic from the assumed distribution. For ease of notation, we define each unbiased plotting position as $p_i = \hat{F}_y\{E[y_{(i)}]\}$; hence,

$$M_i = F^{-1}(p_i) \quad \text{(3)}$$

Probability-Plot Correlation-Coefficient Tests for Normal and Lognormal Hypotheses

Joiner and Rosenblatt (1971) provide an approximation to the inverse of the standard normal distribution. Their approximation

$$M_i = 4.91[p_i^{0.14} - (1 - p_i)^{0.14}] \quad \text{(4)}$$

is used here and in the MINITAB[1] statistical package (Minitab Inc. 1986); more accurate approximations are available (Odeh and Evans 1974). (Note: Use of trade names is for identification purposes only and does not constitute endorsement by the U.S. Geological Survey.) An approximation to the unbiased plotting position for the normal distribution suggested by Cunnane (1978) and developed by Blom (1958) is

$$p_i = \frac{i - \frac{3}{8}}{n + \frac{1}{4}} \quad \text{(5)}$$

An estimate of the PPCC test statistic for a sample y_i, $i = 1, n$ which is hypothesized to be normal is found by substitution of Eqs. 4 and 5 into Eq. 1. Analogously, the PPCC test statistic for a sample x_i, $i = 1, n$, which is hypothesized as two-parameter lognormal, is found by substituting Eqs. 4

and 5 into Eq. 1 using $y_i = \ln(x_i)$. Theoretical percentage points of the distribution of an estimate of r when y_i is normally distributed may be found in Looney and Gulledge (1985) for $n = 1, 100$. Those percentage points are almost identical to the percentage points in Filliben (1975) and Vogel (1986) who used a biased plotting position.

A Probability-Plot Correlation-Coefficient Test for Uniform Distribution

The cumulative probability associated with the value of any random variable is distributed uniformly over the interval [0, 1]. This fact is used later to develop regional hypothesis tests for low-flow frequency distributions in Massachusetts. In general, a uniform random variable, U, over the interval $[a, b]$ has probability density function:

$$f_u(u) = \frac{1}{b - a} \qquad \text{if } a \le u \le b \tag{6a}$$

$$f_u(u) = 0 \qquad \text{otherwise} \tag{6b}$$

and cumulative distribution function (CDF):

$$F_u(u) = 0 \qquad u < a \tag{7a}$$

$$F_u(u) = \frac{u - a}{b - a} \qquad a \le u \le b \tag{7b}$$

$$F_u(u) = 1 \qquad u > b \tag{7c}$$

In this case, the CDF is easily inverted to obtain

$$u = F_u^{-1}[F_u(u)] = a + (b - a)F_u(u) \tag{8}$$

Again, defining $p_i = \hat{F}_u\{E[u_{(i)}]\}$ to be the unbiased plotting position which in this case is the well-known Weibull plotting position $p_i = i/(n + 1)$, we obtain:

$$M_i = a + (b - a)\left(\frac{i}{n + 1}\right) \tag{9}$$

Although the Weibull plotting position is perhaps the most widely used plotting position in hydrology, it is only unbiased when the random variable is uniformly distributed (Cunnane 1978).

For testing the uniform hypothesis, the test statistic is given by Eq. 1 with M_i obtained from Eq. 9 and the $y_{(i)} = u_{(i)}$. This PPCC test statistic is invariant to the distribution parameters a and b. One can show that:

$$r = \frac{\operatorname{cov}\left[u_{(i)}, \frac{i}{n + 1}\right]}{\sqrt{\operatorname{Var}[u_{(i)}] \operatorname{Var}\left(\frac{i}{n + 1}\right)}} \tag{10}$$

which does not depend on the assumed parameters of the uniform distri-

TABLE 1. Critical Points of 1,000 (1 − $\hat{r}$) Where $\hat{r}$ Is the Uniform Probability-Plot Correlation Coefficient

n	Significance Levels								
	0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
10	124.0	81.0	64.0	42.7	27.3	17.4	11.7	9.18	5.79
15	83.6	56.3	44.4	30.2	19.6	12.8	8.94	7.21	4.87
20	63.3	42.4	34.0	23.1	15.2	10.1	7.09	5.80	4.02
25	50.9	34.2	27.4	18.8	12.4	8.29	5.88	4.82	3.39
30	43.2	28.8	23.1	15.8	10.5	7.03	5.02	4.13	2.95
35	36.8	24.8	19.9	13.6	9.04	6.10	4.36	3.62	2.55
40	32.1	21.8	17.5	12.0	7.98	5.40	3.88	3.21	2.30
45	28.7	19.4	15.5	10.7	7.12	4.81	3.47	2.88	2.08
50	25.9	17.6	14.1	9.71	6.43	4.37	3.15	2.61	1.90
55	23.8	15.9	12.8	8.80	5.87	3.98	2.87	2.40	1.72
60	21.7	14.6	11.7	8.09	5.39	3.66	2.65	2.21	1.61
65	20.0	13.4	10.8	7.48	4.99	3.39	2.46	2.05	1.49
70	18.4	12.5	10.0	6.96	4.65	3.17	2.30	1.91	1.39
75	17.3	11.8	9.45	6.53	4.35	2.96	2.15	1.79	1.30
80	16.4	11.0	8.86	6.11	4.07	2.78	2.01	1.67	1.23
90	14.4	9.84	7.92	5.45	3.64	2.47	1.80	1.49	1.09
100	13.2	8.78	7.09	4.91	3.27	2.24	1.63	1.36	0.995
200	6.58	4.45	3.58	2.47	1.65	1.13	0.824	0.688	0.503
500	2.61	1.77	1.43	0.993	0.666	0.457	0.333	0.280	0.205
1,000	1.32	0.888	0.714	0.494	0.333	0.229	0.167	0.140	0.104

Note: This table is based upon 100,000 replicate experiments for all values of n. An example documents the use of this table. The fifth percentage point of $\hat{r}$ when $n = 50$ is determined from $\hat{r}_{0.05} = 1 - 17.6 \times 10^{-3} = 0.982$. Interpolation of the critical points may be accomplished by noting that ln (n) and ln [1,000(1 − $\hat{r}$)] are approximately linearly related for each significance level.

bution. This result applies to any PPCC test statistic for a one- or two-parameter distribution that exhibits a fixed shape.

Because critical points of this test statistic are unavailable in the literature, percentage points (or significance levels) were computed for sample sizes in the range n = 10 to 1,000. This was accomplished by generating 100,000 sequences of uniform random variables each of length n and applying Eqs. 9 and 1 to obtain 100,000 corresponding estimates of r denoted $\hat{r}_i$, $i = 1, \ldots, 100{,}000$. Critical points of the distribution of $\hat{r}$ were obtained by using the empirical sampling procedure

$$r_p = r_{(100{,}000p)} \quad \ldots \quad (11)$$

where r_p denotes the pth quantile of the distribution of $\hat{r}$ and $r_{(100{,}000p)}$ denotes the 100,000p largest observation in the sequence of 100,000 generated values of $\hat{r}$. For large significance levels, p, and large samples, n, the percentage points of the distribution of $\hat{r}$ approach unity and become indistinguishable from that value. Therefore, it is more convenient to tabulate the percentage points of 1,000 (1 − $\hat{r}$). The results of these experiments are summarized in Table 1.

A Probability-Plot Correlation-Coefficient Test for Two-Parameter Weibull Distribution

Since its introduction to the water-resources literature by Gumbel (1954), the extreme value type III distribution, commonly referred to as the Weibull distribution, is considered a theoretically plausible distribution for low flows, much as the extreme value type I or Gumbel distribution is considered a theoretically plausible distribution for floodflows. The CDF of a three-parameter Weibull (W3) random variable, w, takes the form

$$F_w(w) = 1 - \exp\left[-\left(\frac{w-\epsilon}{\nu-\epsilon}\right)\right]^k \quad (12)$$

where the parameters ϵ, ν, and k must be estimated from a sample of streamflows. Two forms of the Weibull (extreme value type III) distribution exist, one corresponding to the distribution of the maximum of many values and another (Eq. 12) corresponding to the distribution of the minimum of many values [see Benjamin and Cornell (1970), pp. 283–284, for a discussion]. The three-parameter Weibull CDF may be expressed in its inverse form as:

$$\ln(w-\epsilon) = \ln(\nu-\epsilon) + \frac{1}{k}\ln\{-\ln[1-F_w(w)]\} \quad (13)$$

Here, one observes that $\ln(w-\epsilon)$ is linearly related to $\ln\{-\ln[1-F_w(w)]\}$; hence, a probability plot is constructed by plotting these two variables against each other. This requires an estimate of the lower bound ϵ; thus, a hypothesis test that does not depend on the distribution's parameters cannot be developed for the W3 distribution. However, if one sets ϵ equal to zero, we obtain the two-parameter Weibull (W2) distribution for which we can construct a PPCC hypothesis test that does not depend on the parameter-estimation procedure.

For the W2 distribution, the PPCC test statistic is again defined by Eq. 1 using $y_{(i)} = \ln(w_{(i)})$ and using:

$$M_i = F_w^{-1}(p_i) = \ln(\nu) + \frac{1}{k}\ln[-\ln(1-p_i)] \quad (14)$$

where p_i = Gringorten's (1963) plotting position for the Gumbel distributions:

$$p_i = \frac{i-0.44}{n+0.22} \quad (15)$$

If a random variable g has a Gumbel distribution, then a Gumbel probability plot produces a linear relation between $g_{(i)}$ and $\ln[-\ln(p_i)]$ (Vogel 1986). From Eqs. 13 and 14, it is clear that a Weibull (W2) probability plot is analogous to a Gumbel probability plot because one plots $\ln[w_{(i)}]$ as a function of $\ln[-\ln(1-p_i)]$. Hence, Gringorten's plotting position is appropriate for either distribution. Unbiased plotting positions for the generalized extreme value distribution (Arnell et al. 1986) depend on the shape parameter of the distribution, k; thus, they are not suitable for use in a distributional hypothesis test. The W2 PPCC test is invariant to the fitting procedure used to estimate ν and k in Eq. 14. The proof is similar to Eq. 10 here and Eq. 11 in Vogel (1986).

TABLE 2. Critical Points of 1,000 (1 − $\hat{r}$) where $\hat{r}$ Is Two-Parameter Weibull Probability-Plot Correlation Coefficient

	Significance Levels								
	0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	132.0	90.9	73.8	49.6	31.9	20.0	12.7	9.66	
	107.0	72.6	58.0	39.4	25.6	16.5	11.0	8.65	
	97.2	61.6	48.9	33.3	21.7	14.2	9.71	7.68	
	86.0	54.0	42.3	28.7	18.9	12.6	8.70	6.97	
	77.1	47.5	37.5	25.5	16.8	11.2	7.85	6.32	
	71.9	44.1	34.4	23.2	15.3	10.2	7.14	5.83	
	66.6	40.1	31.3	21.1	13.9	9.28	6.59	5.36	
	62.6	37.0	28.7	19.3	12.8	8.64	6.12	4.97	
	60.1	35.3	27.2	18.2	12.0	8.06	5.71	4.71	
	55.5	33.1	25.5	17.0	11.3	7.61	5.40	4.42	
	53.2	30.7	23.8	16.1	10.6	7.18	5.14	4.21	
	51.1	29.1	22.5	15.1	9.99	6.80	4.86	4.00	
	48.2	27.8	21.4	14.4	9.53	6.45	4.62	3.78	
	48.1	26.7	20.5	13.7	9.06	6.11	4.40	3.62	
	44.8	25.8	19.7	13.2	8.63	5.87	4.20	3.47	
	42.1	23.4	17.9	11.9	7.93	5.41	3.88	3.20	
	39.4	22.3	16.9	11.3	7.41	5.06	3.62	2.99	
	24.1	13.2	10.1	6.68	4.43	3.04	2.22	1.86	
	12.2	6.60	4.96	3.30	2.20	1.52	1.11	0.93	
	6.73	3.79	2.82	1.90	1.27	0.88	0.64	0.54	

Note: This table is based upon 50,000 replicate experiments except for the cases n = 500 and 1,000 for which only 10,000 replicate experiments were performed. The fifth percentage point of $\hat{r}$ when $n = 50$ is determined from $\hat{r}_{0.05} = 1 - 35.3 \times 10^{-3} = 0.9647$. Interpolation of the critical points may be accomplished by noting that ln (n) and ln [1,000(1 − $\hat{r}$)] are approximately linearly related for each significance level.

Because critical points of this test statistic are unavailable in the literature, percentage points (or significance levels) were computed for sample sizes in the range n = 10 to 1,000. This was accomplished by generating 50,000 sequences of W2 random variables each of length n and applying Eqs. 14, 15, and 1 to obtain 50,000 corresponding estimates of $\hat{r}$. Critical points of the distribution of r were obtained by use of the empirical sampling procedure described earlier for the uniform PPCC test, except that 50,000 experiments were performed for each value of n. The percentage points of the distribution of 1,000 (1 − $\hat{r}$) are summarized in Table 2 for the Weibull distribution.

Low-Flow Frequency Hypothesis Tests in Massachusetts

The tests introduced in the previous section for the LN2 and W2 distributions are employed here to test the hypotheses that annual minimum seven-day, low flows in Massachusetts arise from each of those distributions. Twenty-three of the U.S. Geological Survey's streamflow-gaging stations in Massachusetts with the following attributes were selected:

TABLE 2. Critical Points of 1,000 (1 − $\hat{r}$) where $\hat{r}$ Is Two-Parameter Weibull Probability-Plot Correlation Coefficient

	Significance Levels								
n (1)	0.01 (2)	0.05 (3)	0.10 (4)	0.25 (5)	0.50 (6)	0.75 (7)	0.90 (8)	0.95 (9)	0.99 (10)
10	132.0	90.9	73.8	49.6	31.9	20.0	12.7	9.66	5.80
15	107.0	72.6	58.0	39.4	25.6	16.5	11.0	8.65	5.59
20	97.2	61.6	48.9	33.3	21.7	14.2	9.71	7.68	5.09
25	86.0	54.0	42.3	28.7	18.9	12.6	8.70	6.97	4.75
30	77.1	47.5	37.5	25.5	16.8	11.2	7.85	6.32	4.24
35	71.9	44.1	34.4	23.2	15.3	10.2	7.14	5.83	3.98
40	66.6	40.1	31.3	21.1	13.9	9.28	6.59	5.36	3.70
45	62.6	37.0	28.7	19.3	12.8	8.64	6.12	4.97	3.42
50	60.1	35.3	27.2	18.2	12.0	8.06	5.71	4.71	3.24
55	55.5	33.1	25.5	17.0	11.3	7.61	5.40	4.42	3.15
60	53.2	30.7	23.8	16.1	10.6	7.18	5.14	4.21	2.96
65	51.1	29.1	22.5	15.1	9.99	6.80	4.86	4.00	2.78
70	48.2	27.8	21.4	14.4	9.53	6.45	4.62	3.78	2.69
75	48.1	26.7	20.5	13.7	9.06	6.11	4.40	3.62	2.54
80	44.8	25.8	19.7	13.2	8.63	5.87	4.20	3.47	2.46
90	42.1	23.4	17.9	11.9	7.93	5.41	3.88	3.20	2.27
100	39.4	22.3	16.9	11.3	7.41	5.06	3.62	2.99	2.11
200	24.1	13.2	10.1	6.68	4.43	3.04	2.22	1.86	1.35
500	12.2	6.60	4.96	3.30	2.20	1.52	1.11	0.93	0.69
1,000	6.73	3.79	2.82	1.90	1.27	0.88	0.64	0.54	0.41

Note: This table is based upon 50,000 replicate experiments except for the cases n = 500 and 1,000 for which only 10,000 replicate experiments were performed. The fifth percentage point of $\hat{r}$ when n = 50 is determined from $\hat{r}_{0.05} = 1 - 35.3 \times 10^{-3} = 0.9647$. Interpolation of the critical points may be accomplished by noting that ln (n) and ln [1,000(1 − $\hat{r}$)] are approximately linearly related for each significance level.

Because critical points of this test statistic are unavailable in the literature, percentage points (or significance levels) were computed for sample sizes in the range n = 10 to 1,000. This was accomplished by generating 50,000 sequences of W2 random variables each of length n and applying Eqs. 14, 15, and 1 to obtain 50,000 corresponding estimates of $\hat{r}$. Critical points of the distribution of r were obtained by use of the empirical sampling procedure described earlier for the uniform PPCC test, except that 50,000 experiments were performed for each value of n. The percentage points of the distribution of 1,000 (1 − $\hat{r}$) are summarized in Table 2 for the Weibull distribution.

Low-Flow Frequency Hypothesis Tests in Massachusetts

The tests introduced in the previous section for the LN2 and W2 distributions are employed here to test the hypotheses that annual minimum seven-day, low flows in Massachusetts arise from each of those distributions. Twenty-three of the U.S. Geological Survey's streamflow-gaging stations in Massachusetts with the following attributes were selected:

TABLE 4. Statistics of Significance Levels Associated with Two-Parameter Lognormal and Two-Parameter Weibull Distributional Hypotheses

Regional hypothesis (1)	SIGNIFICANCE LEVELS: Mean, Theoretical (2)	Mean, Observed (3)	Standard Deviation, Theoretical (4)	Standard Deviation, Observed (5)	$\hat{r}$ (6)	Regional significance level (7)
LN2	0.5	0.42	0.289	0.30	0.983	0.45
W2	0.5	0.21	0.289	0.25	0.886	0.005

null hypothesis, one must reject one site under the LN2 null hypothesis and ten sites under the W2 null hypothesis. Under either null hypothesis one would expect to reject 5% of the sites if the type I error is 5%. Therefore, one would expect one rejection among 23 sites, as is the case under the LN2 hypothesis. Clearly, 10 rejections under the W2 hypothesis are unacceptable.

A Regional-Hypothesis Test

The significance levels corresponding to each hypothesis test at each site in Table 3 are simply cumulative probabilities associated with the distributions of $\hat{r}$. Such significance levels are, in theory, uniformly distributed over the interval [0, 1], if the sites are considered to be independent. Thus, under the null hypothesis, the 23 values of significance levels corresponding to the LN2 and W2 hypotheses in Table 3 should have sample mean and standard deviations approximately equal to 0.5 and 0.289, respectively. Table 4 illustrates that the observed values are very close to the theoretical values under the LN2 regional hypothesis but not under the W2 hypothesis.

A regional hypothesis test may be performed by applying a uniform PPCC hypothesis test on the significance levels in Table 3, assuming that there are 23 independent samples (sites). The values of $\hat{r}$ in Table 4 are computed using Eqs. 1 and 9 with $y_{(i)}$ in Eq. 1 equal to the significance levels in Table 3. The regional significance levels associated with the LN2 and W2 alternatives are then determined using the uniform PPCC test statistic in Table 1. If one accepts a 5% type I error probability, one must reject the W2 regional hypothesis but not the LN2 regional hypothesis.

The regional hypothesis test makes it quite evident that the low flows are poorly approximated by the W2 hypothesis on a regional basis. Yet, if one were to simply perform a W2 hypothesis test site-by-site, allowing a type I error probability of 1%, then the W2 hypothesis would only be rejected at four sites. Such site-by-site hypothesis testing can be misleading; the regional-hypothesis test described here is favored.

Goodness of Fit of Low-Flow Frequency Distributions

Filliben (1975) and Vogel (1986) have recommended the use of the PPCC test statistic for comparing the goodness of fit of competing distributions and parameter-estimation procedures. In this section, the goodness of fit of the LN3, W3, and LP3 probability distributions are compared using the PPCC test statistic. Because computation of the PPCC test statistic for three-parameter distributions requires estimation of at least one of the distribution's parameters, regional hypothesis tests cannot be performed for three-parameter alternatives. Use of any of the critical points of the test statistic $\hat{r}$ provided here for testing a three-parameter distribution will lead to fewer rejections of the null hypothesis than one would anticipate. This is because only two parameters are estimated in the construction of the PPCC tests developed here and elsewhere, yet three parameters are required to fit the LN3, LP3, and W3 distributions. For this reason, hypothesis tests are not used for three-parameter alternatives. Instead, the PPCC test statistic is only used as a goodness-of-fit statistic to discriminate among alternative parameter-estimation procedures and competing probability models.

Probability Plots for Lognormal Distribution

Three methods of fitting a lognormal distribution to annual minimum seven-day low flows in Massachusetts are compared by analytically constructing a probability plot using Eqs. 1, 4, and 5. Here, the values of $y_{(i)}$ in Eq. 1 are computed as:

$$y_{(i)} = \ln (x_{(i)} - \hat{\tau}) \quad (16)$$

where the $x_{(i)}$ are the ordered annual minimum seven-day low flows and $\hat{\tau}$ is an estimate of the lower bound of the low flows. For the two-parameter lognormal distribution, τ is set equal to zero and the procedure is termed LN2. For the LN2 distribution, one obtains identical estimates of $\hat{r}$ using method of moments (MM) estimators or maximum likelihood estimators (MLE) because $\hat{r}$ is invariant to the parameters of the distribution. For the three-parameter lognormal distribution, two procedures are compared. Matalas (1963), Condie and Nix (1975), and Stedinger (1980) described the method of moments procedure, which we term LN3-MM. Stedinger recommended the use of a quantile lower bound estimator, which we term LN3-STLB (Stedinger 1980, Eq. 20; or Loucks et al. 1981, Eq. 3.60). Because the LN3-MM and LN3-STLB procedures for estimating τ in Eq. 16 are well documented in the literature, we do not repeat those equations here.

Fig. 1 illustrates the values of $\hat{r}$ for each of the 23 sites described in Table 3 using the LN2, LN3-MM, and LN3-STLB procedures. In general, the LN2 and LN3-STLB procedures produce much more linear probability plots than the LN3-MM procedure, as is evidenced from the larger values of $\hat{r}$ associated with those procedures (to improve the presentation of the results, the sites are arbitrarily ordered by increasing values of $\hat{r}$ associated with the LN2 PPCC test). On the basis of these results, it is not surprising that Matalas (1963) and Condie and Nix (1975) rejected the LN3-MM procedure for fitting low-flow frequency distributions. However, rejection of the LN3-MM procedure need not imply rejection of the LN3 distribution. The LN3-MM procedure requires a sample estimate of the skew coefficient to obtain an estimate of τ in Eq. 16; such sample estimates are known to be highly unstable (Wallis et al. 1974). Both the LN2 and LN3-STLB procedures provide an excellent fit to the distribution of low flows in Massachusetts. In fact, the LN3-STLB procedure always resulted in values of $\hat{r}$ greater than 0.976, except at site 2, where Stedinger's lower bound could not be computed.

Probability Plots for Weibull Distribution

Three methods of fitting a three-parameter Weibull distribution (W3) to annual minimum seven-day low flows in Massachusetts are compared using

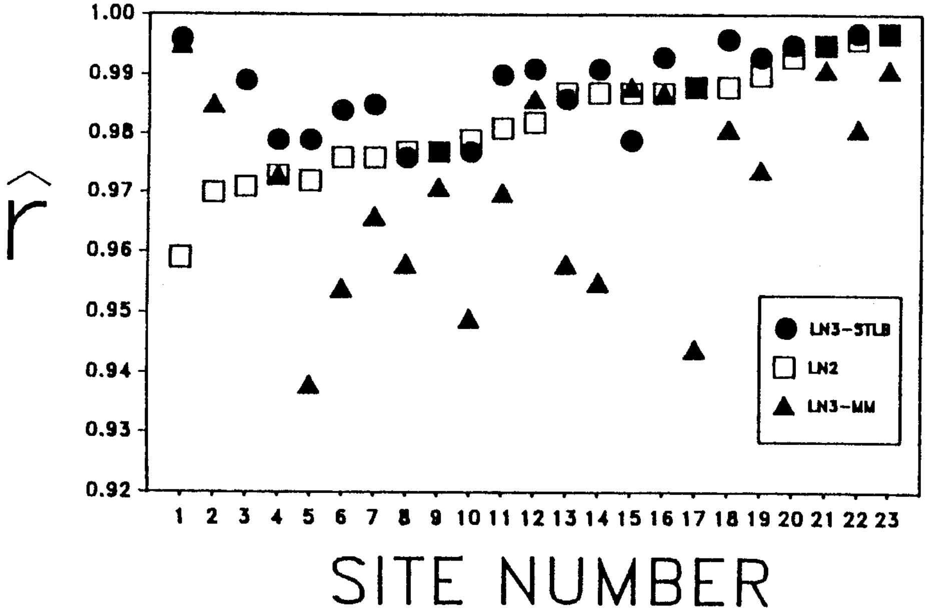

FIG. 1. Probability-Plot Correlation-Coefficient Test Statistics Corresponding to LN2, LN3-MM, and LN3-STLB Procedures

the PPCC test statistic. The PPCC test statistic is defined in Eq. 1 with the $y_{(i)}$ given by:

$$y_{(i)} = \ln [w_{(i)} - \hat{\epsilon}] \quad (17)$$

in which $w_{(i)}$ = the ordered annual minimum seven-day low flows; and $\hat{\epsilon}$ = an estimate of the lower bound of the low flows. Reasonable estimates of ϵ must be less than the smallest observation $w_{(1)}$, otherwise Eq. 17 is undefined. The M_i in Eq. 1 are defined as:

$$M_i = \ln [-\ln (1 - p_i)] \quad (18)$$

where the p_i are obtained from Eq. 15. Analogous to the lognormal distribution, each of the fitting procedures for the W3 distribution amount to different approaches for estimating the lower bound ϵ. Tables 3 and 4 documented that the W2 procedure failed a regional hypothesis test; hence, that procedure is dropped from consideration here. The approaches considered here for fitting a W3 distribution are: (1) The method of moments (W3-MM); (2) the method of maximum likelihood (W3-MLE); and (3) the method of smallest observed drought (W3-MSOD). Gumbel (1954, 1958), Matalas (1963), Deininger and Westfield (1969), Condie and Nix (1975), and Kite (1977) describe the WE-MM procedure. Cohen (1975), Condie and Nix (1975), and Kite (1977) describe the W3-MLE procedure. Gumbel (1963), Deininger and Westfield (1969), Condie and Nix (1975), and Kite (1977) describe the W3-MSOD procedure. Because all three procedures are fully described elsewhere, the equations are not repeated here. Each of these procedures were used to estimate the lower bound ϵ in Eq. 17, and corresponding estimates of $\hat{r}$ were obtained by substitution of Eqs. 17 and 18 into Eq. 1, where the p_i are obtained from Eq. 15. The estimates of r are summarized in Fig. 2

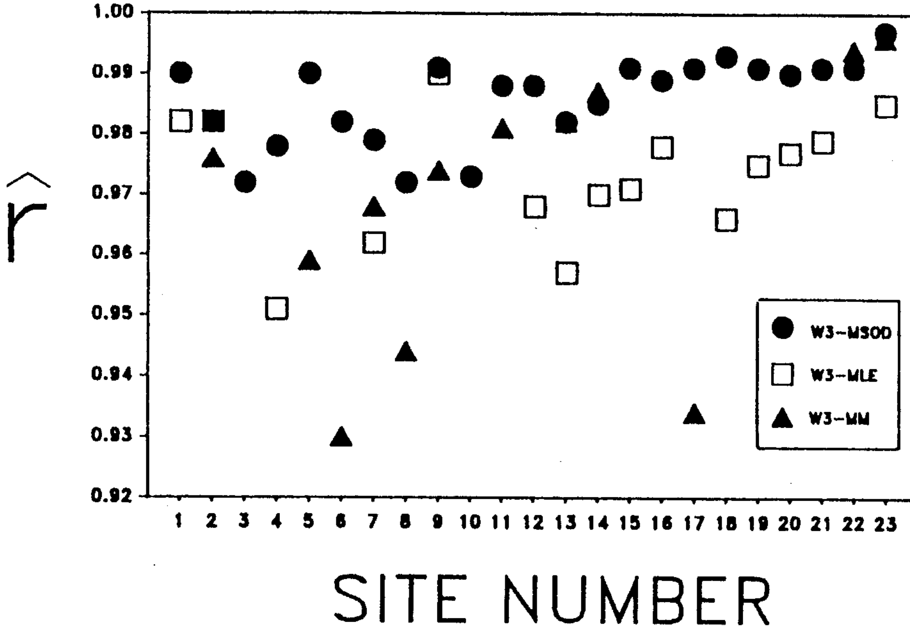

FIG. 2. Probability-Plot Correlation-Coefficient Test Statistics Corresponding to W3-MM, W3-MLE, and W3-MSOD Procedures

corresponding to these three-parameter estimation procedures. Overall, the W3-MSOD procedure resulted in higher values of $\hat{r}$ (and thus more linear probability plots) than the W3-MM and W3-MLE procedures. Furthermore, reasonable estimates of $\hat{r}$ were obtained at all 23 sites using the W3-MSOD procedure, whereas many of the other procedures failed to produce feasible estimates of the parameters of the distribution. For example, the W3-MM procedure generated values of the lower bound, $\hat{\epsilon}$, which were greater than the smallest observation, $w_{(1)}$, at nine out of 23 sites (or roughly 39% of the sites). Similarly, Matalas (1963) found $\hat{\epsilon} > w_{(1)}$ using the W3-MM procedure at ten out of 34 streams (roughly 29% of the sites) in his investigation.

The W3-MLE procedure often fails to provide feasible parameter values. The W3-MLE procedure requires the simultaneous solution of three nonlinear equations in ϵ, ν, and k. Because the parameter ν can be removed from two of those equations, the procedure reduces to the solution of two nonlinear equations in ϵ and k which must be solved simultaneously. The W3-MLE procedure did not converge, and, hence, no feasible solution (for example, $\hat{\epsilon} > w_{(i)}$ existed at eight of 23 sites (roughly 35% of the sites). As in the study by Loganathan et al. (1985), the W3-MLE procedure failed to converge at 14 out of 20 sites (70% of the sites) in their investigation.

Condie and Nix (1975) provide the following recommendations for fitting a W3 distribution:

> Maximum likelihood should be used in the first instance to estimate the parameters. If a lower boundary parameter is not found within the range zero to the minimum of the data series, then try estimating parameters by the method of the smallest observed drought, and finally by moments.

Their algorithm is based principally on the ability of each parameter estimation procedure to generate values of $\hat{\epsilon}$ in the interval $[0, w_{(1)}]$. In this study, negative estimates of ϵ are allowed, and the objective is to obtain the most linear probability plot (highest value of $\hat{r}$). Fig. 2 indicates that even when the W3-MLE procedure converged, corresponding estimates of $\hat{r}$ were usually significantly lower than for the W3-MSOD procedure. Hence, the W3-MSOD is clearly preferred in this study because it always leads to feasible solutions. The W3-MLE procedure requires a relatively complex numerical algorithm; our algorithm was verified by testing it on the sample of 100 observations from a Weibull population given by Dubey (1967, Table 1), as is common practice in the field of statistics [for example, Wingo (1972), pp. 91–92].

On the basis of the results in Fig. 2, we favor the W3-MSOD procedure over either the W3-MM or W3-MLE procedures in Massachusetts because it always led to feasible solutions, it produced consistently linear probability plots, and it is computationally convenient compared to the W3-MLE procedure. Fig. 2 also illustrates that the PPCC test statistic is a useful tool for discriminating among alternative parameter-estimation procedures.

Probability Plots for Log-Pearson Type III Distribution

The log-Pearson type III (LP3) distribution is used widely by hydrologists for modeling floodflow frequencies. The Interagency Advisory Committee on Water Data ("Guidelines" 1982) recommends fitting the LP3 distribution by applying the method of moments for the P3 distribution to the logarithms of annual maximum floodflows. Their recommendations are based on Beard's (1974) comprehensive study of flood flows. No comprehensive studies of the type performed by Beard have been performed for low flows. Nevertheless, Loganathan et al. (1986, p. 131) and Tasker (1987, p. 1077) state that the LP3 distribution is used widely to model low-flow frequencies. In fact the U.S. Geological Survey's WATSTORE data retrieval and analysis system (Hutchinson 1975) routinely fits an LP3 distribution to low flows by using the method of moments in log space for the P3 distribution (LP3-MM). To confirm the suspicion that the LP3-MM procedure provides a reasonable model of low-flow frequencies, we favor the use of probability plots and the PPCC test statistic. Loucks et al. (1981, p. 116) recommend the use of the Wilson-Hilferty transformation for constructing a probability plot for the three-parameter gamma distribution. We use the Wilson-Hilferty transformation to construct probability plots for the LP3 distribution and to estimate the corresponding PPCC statistic.

For the LP3 distribution, the PPCC test statistic is defined in Eq. 1 with the $y_{(i)} = \ln [x_{(i)}]$ where the $x_{(i)}$ are the ordered, observed annual minimum seven-day low flows. The M_i in Eq. 1 are determined using

$$M_i = m_y + s_y\left\{\frac{2}{G}\left[1 + \frac{G\Phi^{-1}(p_i)}{6} - \frac{G^2}{36}\right]^3 - \frac{2}{G}\right\} \quad (19)$$

in which m_y, s_y, and G = sample estimates of the mean, standard deviation, and skew coefficient of the $y_{(i)}$; and $\Phi^{-1}(p_i)$ = the inverse of a standard normal random variable which is obtained by substitution of Eq. 5 into Eq. 4. The Wilson-Hilferty transformation gives quantiles of a gamma distri-

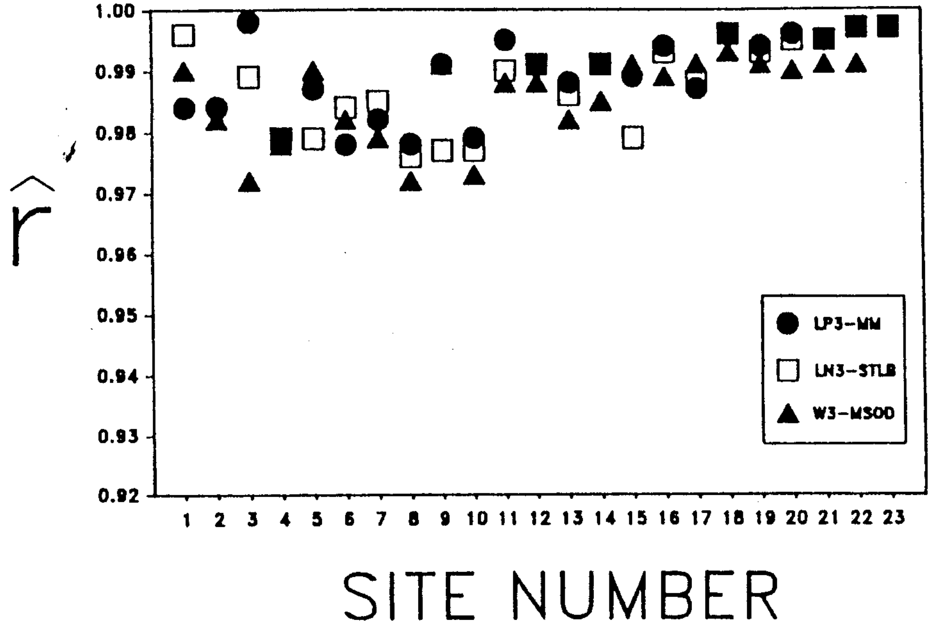

FIG. 3. Probability-Plot Correlation-Coefficient Test Statistics Corresponding to LP3-MM, LN3-STLB, and W3-MSOD Procedures

bution in terms of quantiles of a standard normal distribution; hence, Blom's plotting position (Eq. 5) is used because it is unbiased for the normal distribution. Eq. 19 provides an accurate approximation to the inverse of the LP3 distribution as long as $-3 \leq G \leq 3$, which is the case here. When the skew coefficient G lies outside this interval, Kirby's (1972) modified transformation may be used instead of Eq. 19, although even Kirby's modified transformation fails to reproduce the marginal distribution for large values of G and/or if the flows exhibit significant autocorrelation (Obeysekera and Yevjevich 1985). An unbiased estimator of the skew coefficient G is employed here which corresponds to method 9 in Bobee and Robitaille (1977) and the indirect method of moments in Kite (1977, p. 126). Lettenmaier and Burges (1980) provide alternative bias correction factors for estimators of both the standard deviation and skew coefficient of the LP3 distribution; in fact a variety of alternative procedures are available, as is evidenced in a study by Bobee and Robitaille (1977) of 11 competing parameter-estimation procedures for the LP3 distribution. Our goal here is to simply compare the fit of the LP3 distribution with alternative three-parameter distributions; hence, we only consider this one parameter-estimation procedure.

Fig. 3 compares the values of $\hat{r}$ obtained using the three fitting procedures: LP3-MM, W3-MSOD, and LN3-STLB. Fig. 3 illustrates that the LP3-MM procedure resulted in more nearly linear probability plots than the W3-MSOD and LN3-STLB procedures at 17 out of 23 sites (74% of the sites). Similarly, the LN3-STLB procedure resulted in more nearly linear probability plots than the W3-MSOD procedure at 17 out of 23 sites.

Figs. 4 and 5 illustrate probability plots at two sites (sites 3 and 6) on lognormal probability paper. Here the circles are obtained by plotting the annual minimum seven-day low flows as a function of the M_i obtained from

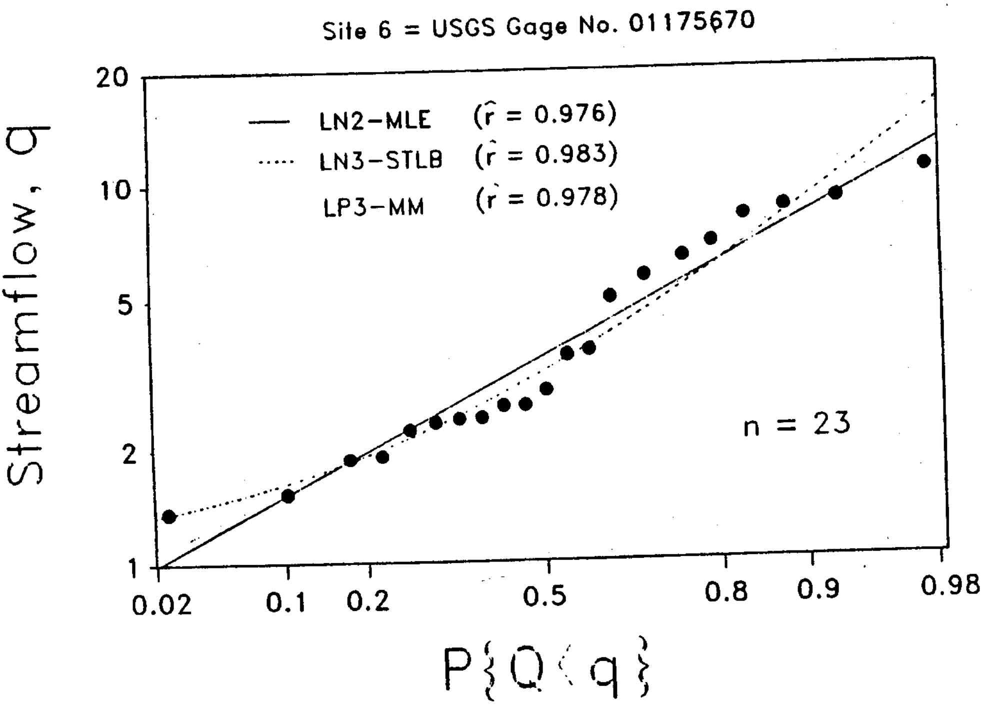

FIG. 4. Comparison of LN2-MLE, LN3-STLB, and LP3-MM Probability Plots at Site Number 6

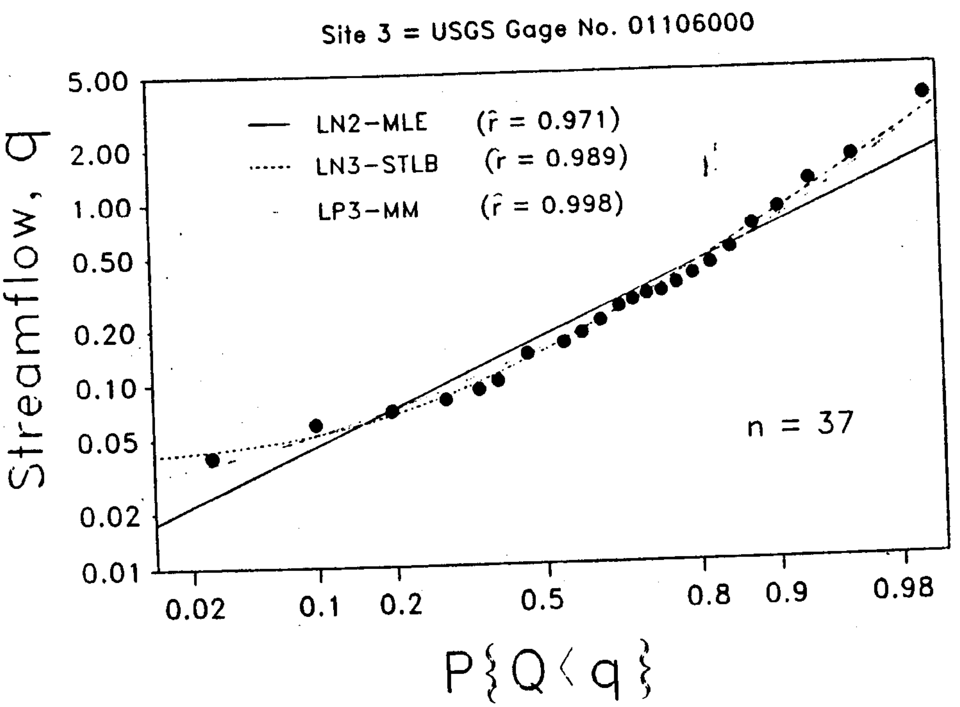

FIG. 5. Comparison of LN2-MLE, LN3-STLB, and LP3-MM Probability Plots at Site Number 3

Eqs. 4 and 5. The probability plots corresponding to the LN2-MLE, LN3-STLB, and LP3-MM procedures also are illustrated along with their corresponding PPCC values. As shown in Fig. 1, sites 3 and 6 resulted in relatively low values of $\hat{r}$ for the LN2 procedure compared with the other sites. Figs. 4 and 5 provide a graphical representation of the differences between these three fitting procedures and the corresponding values of $\hat{r}$. At most of the other sites, the differences among the $\hat{r}$ are smaller than evidenced in Figs. 4 and 5, in which case the three procedures are almost indistinguishable.

All of the procedures in Fig. 3 produced probability plots with values of $\hat{r}$ in excess of 0.97. Figs. 1–3 suggest that the LN2, LN3-STLB, W3-MSOD, and LP3-MM procedures all provide reasonable approaches for fitting low-flow frequency distributions in Massachusetts.

Model Choice

When fitting a hypothesized distribution function to observed data, and plotting the resulting cumulative distribution function using the procedures described here, one anticipates a linear probability plot. Yet, a linear probability plot only provides sample evidence in our quest to prove the null hypothesis. The design engineer also must consider the sampling properties of the quantile estimate, which, in this instance, is $\hat{Q}_{7,10}$.

Fig. 6 compares the estimates, $\hat{Q}_{7,10}$, using the LN2-MLE and LP3-MM procedures. The maximum likelihood procedure (LN2-MLE) is used to fit the LN2 distribution because it was recommended by Stedinger (1980). As anticipated from the linearity of their corresponding probability plots, both procedures generate almost identical estimates of $Q_{7,10}$ at all 23 sites.

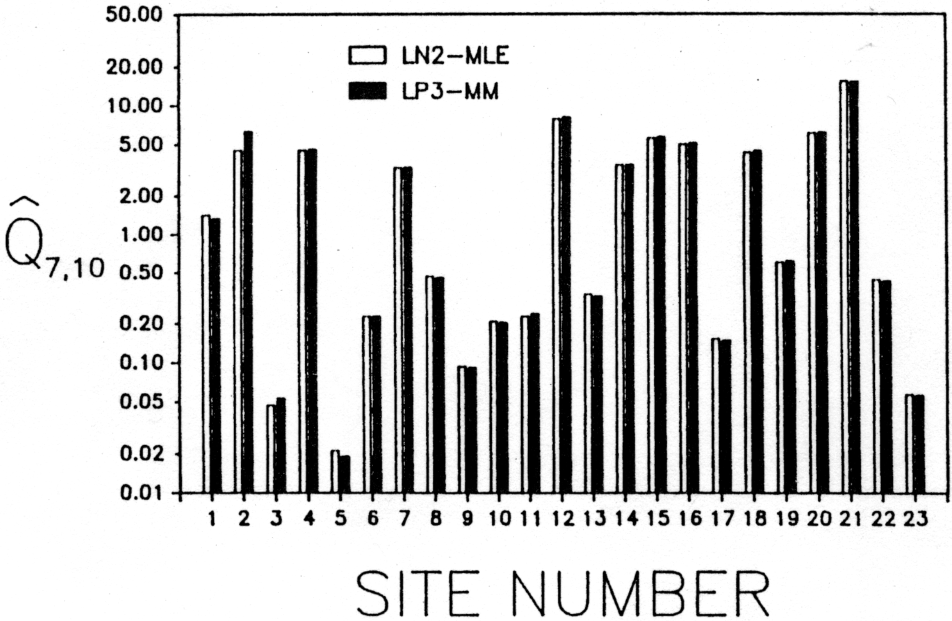

FIG. 6. Comparison of Estimates of Seven-Day Ten-Year Low-Flow Statistic Using LN2-MLE and LP3-MM Procedures

A tradeoff exists between the linearity of each probability plot and the precision (i.e., root mean square error, RMSE) of corresponding estimates of $Q_{7,10}$. Clearly three- or even four-parameter distributions can generate more linear probability plots than a two-parameter distribution, at the expense of additional sampling error in the parameters. For example, the LP3-MM procedure requires estimation of a third parameter, G, not required by the LN2-MLE procedure, and estimates of G are known to be highly unstable for most of the sample sizes encountered in this study (see Wallis et al. 1974). Hence one may anticipate that estimation of a third or even fourth parameter could produce quantile estimates with higher RMSE than a two-parameter alternative, particularly in situations when the sample size is small and the two-parameter model provides a reasonable fit, as is the case in this study for the LN2 distribution. For example, Stedinger (1980) documents a variety of situations in which the RMSE of quantile estimates resulting from use of the LN2-MLE procedure is lower than the RMSE of quantile estimates resulting from the use of the LN3-STLB procedure when streamflows arise from an LP3 parent.

Quantile estimation is further confounded by the issue of independence. Typically, sequences of annual minimum seven-day low flows exhibit serial correlation which tends to inflate the RMSE of corresponding estimates of $Q_{7,10}$. For example, the average estimate of the first-order serial correlation for the basins of this study is 0.19 with sample estimates which range from −0.12 to 0.56. Tasker (1983) provides a measure of the effect of serial correlation on the effective record length associated with quantile estimates for the LN2 and LP3 distributions.

Conclusions

Probability plots and the probability-plot correlation-coefficient (PPCC) test statistic are introduced for testing alternative low-flow distribution hypotheses and for discriminating among competing parameter-estimation procedures. The PPCC hypothesis test for the uniform distribution derived here is shown to be useful for testing regional distributional hypotheses assuming sites are independent. Regional PPCC hypothesis tests are available for testing the normal, lognormal (LN2), Weibull (W2), and Gumbel two-parameter hypotheses. In addition, this study derives PPCC test statistics for the three-parameter lognormal (LN3), Weibull (W3), and log Pearson type-III (LP3) distributions. PPCC test statistics are computed at 23 unregulated sites in Massachusetts, and the following conclusions have been reached:

1. The regional hypothesis that annual minimum seven-day low flows in Massachusetts arise from a W2 distribution is rejected; however, there is almost no evidence to support the rejection of the LN2 regional hypothesis for the same low flows.
2. The PPCC test statistic was found to be a useful tool for discriminating among competing probability distributions and parameter-estimation procedures. Seven procedures were compared for their ability to generate linear probability plots for low flows. Overall, four of those seven procedures (LN2, LN3-STLB, W3-MSOD, and LP3-MM) consistently produced linear probability plots as measured by the PPCC test statistic. Of those four procedures, the LP3-MM procedure usually performed slightly better than the other procedures. This result

agrees with Tasker's (1987) conclusions for Virginia streams.

3. Previous studies have rejected the lognormal probability distribution because those studies used the method of moments. We found the LN2 and the LN3-STLB procedures to be competitive with other preferred alternatives such as LP3-MM and W3-MSOD. In fact, the LN3-STLB procedure performed slightly better than the W3-MSOD at most sites in Massachusetts. The LN3-MM procedure performed poorly, as it did in previous studies by Matalas (1963) and Condie and Nix (1975), for other locations in the United States and Canada.

4. A complex tradeoff exists between the linearity of a probability plot and the precision of estimates of design quantiles. Probability distributions with three or more parameters tend to produce quite linear probability plots, because additional parameters allow more flexibility in terms of the location, scale, and shape of the modeled distribution. However, this additional flexibility often occurs at the expense of a loss in precision associated with estimated quantiles.

5. The PPCC test statistic, based on information contained in a probability plot, is computationally convenient and is easily extended to other one- and two-parameter distributional hypotheses not reported here. As long as the hypothesized cumulative distribution function of a random variable can be expressed (or approximated) in its inverse form, the PPCC test statistic $\hat{r}$ in Eq. 1 is easily calculated. We hope that future studies will extend the PPCC tests described here to other regions so more general conclusions regarding the distribution of low flows can be reached.

Acknowledgments

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