APM 153 LECTURE TEN - Exponential Growth, Euler's e, Discrete vs. Continuous Calculations.

Exponential Growth

(1) Exponential growth occurs when the rate of growth increases as the amount increases.

(2) One of the easiest examples of exponential growth to understand is the increase of money in a bank account over time as a result of compound interest.

(3) For example, given $100 in a bank account, at the end of one year at 5% interest you would have.

$$\$100 * 1.05 = \$105$$

(4) After two years $\$100 * 1.05 * 1.05 = \$100 * 1.05^2 = 110.25$

(5) And in general, after n years you would have $\$100 * 1.05^{n}$

(6) Exponential growth occurs because the interest paid on your account each year increase as your bank account grows.

Negative Exponents

(7) Alternatively, we can use negative exponents to determine how much money we would need to deposit to obtain a given future amount at 5% interest.

(8) Let's say we want to deposit some money so that in 5 years time, the bank account grows to $1000 dollars at 5% interest. How much money do we have to deposit up front?

(9) In this case, we use a negative exponent to calculate the initial deposit.

$$\$1000 * 1.05^{-5} = \$1000 / 1.05^{5} = 620.92$$

(10) We would have to deposit $620.92 in order to have $1000 at the end of 5 years.

Euler’s e

(11) One of the most useful tools for calculating exponential growth is a number called **Euler’s e**, named for the Swiss mathematician Leonhard Euler (1707-1783).

(12) In addition to, *e*, the base of natural logs (1727), we owe to Euler the notation *f(x)* for a function (1734), the symbol *ð* to represent the ratio of a circle’s circumference to its diameter (1755), and the use of *i* for the square root of -1 (1777).

(13) Euler’s e is defined as

$$e = \left(1 + \frac{1}{\infty}\right) \qquad 2.7182818$$

(14) Euler’s e is a non-terminating, non-repeating decimal which is also known as an *irrational* number. Because it is an irrational number, we can only ever use an **approximation** of the true value of Euler’s e.

(15) We can see how Euler’s e can be used to help calculate exponential functions by returning to our banking example.

(16) If we put $100 in a bank account at 5% interest and receive only one interest payment a year (i.e. “annual” interest), we have the equation,...

$$\$100 * 1.05 = \$105$$

But interest **isn’t** usually calculated only once a year is it?

(17) What would the equation look like if interest were calculated twice a year? At 5% annual interest, we would receive half of our interest at a time, so,..

$$\$100 * 1.025 * 1.025 = \$100 * 1.025^2$$

(18) It is important to note the in the equation above, we receive **interest on our interest**.

(19) Another way of writing the equation would be,...

$$\$100 * \left(1 + \frac{0.05}{2}\right)^2$$

(20) If we received interest 4 times a year we would write the equation as,...

$$\$100 * \left(1+\frac{0.05}{4}\right)^{4}$$

(21) And if we received interest every day, we would write the equation as,...

$$\$100 * \left(1+\frac{0.05}{365}\right)^{365}$$

(22) You will notice that the amount of interest looks a lot like the formula for Euler's e. In fact, we can rewrite the interest using the formula for Euler's e as,...

$$\$100 * \left(1+\frac{1}{365}\right)^{365 * 0.05}$$

or more precisely as,...

$$\$100 * \left(1+\frac{1}{\infty}\right)^{\quad * 0.05}$$

(23) Which means that the exponential equation would be

$$\$100 * e^{0.05}$$

(24) In fact, the general equation for exponential growth is given as,..

$$N_t = N_0 * e^{r * t}$$

Where N_0 = amount at time 0

N_t = amount at time t

r = the rate of growth

t = the length of time (years, months, days, etc.)

(25) In Matlab we can use Euler's e by using the **exp( )** function as in the line of code

$$N_t = N_0 * \exp(r * t)$$

Discrete versus Continuous Calculations

(26) The exponential growth equation from 24 above works really well for some calculations such as the growth of the human population on Earth, or the growth of bacteria in a large reaction vessel.

(27) The exponential growth equation using Euler's e works well because the number of humans or bacteria is so large that it appears that growth is occurring **continuously**.

(28) Some things however, **do not change continuously**. They don't change every second, or even every hour. Instead, things like the amount of money in your bank account changes or the amount of money that you owe on your credit card is calculated **once a day**.

(29) When change occurs only once per time period, we say that the change is **discrete** which is another way of saying "**distinct and separate**".

(30) When change occurs only once an hour, or once a day, or once a year, we say that the change occurs during a discrete **time-step**.

(31) If change occurs once a day, we say that the **time step** is **daily**
once a month the **time step** is **monthly**

once a week the **time step** is ________________

once a year the **time** step is ________________

(32) When change occurs according to a discrete time step, we calculate change using a method that will perform our calculations once every time period.

(33) The method we use to repeat a calculation once every time step is called a **do-loop**. In Matlab the do-loop is also called a "**for**" loop. (read Chapter 4, pages 147-161).

(34) In a do-loop, the program is told to execute a calculation once every time step as in,..

```
for i = 1: 100
   bank = bank + (bank* interest)
end
```